Lovász Local Lemma

In many of the existence proofs that we've looked at so far in this course, our problem was of the form: we have set of "bad" events and we would like to show that with nonzero probability, no bad events happen. The nonzero probability then proves that it is possible for no bad events to happen, i.e. there is a configuration of our model that avoids any bad events. For example: there is a vertex coloring in our graph that avoids monochromatic edges, or there is an edge coloring of K_n that avoids monochromatic subgraphs, etc...

First, we need a slightly more refined definition of independence; one that applies to sets of events rather than just pairs.

Definition 1. *We say that an event A is mutually independent of a set of events S, if*

$$\mathbb{P}(A) = \mathbb{P}(A \mid T),$$

where $T \subseteq \{B \mid B \in S \text{ or } \overline{B} \in S\}$.

Note that mutual independence is not the same as pairwise independence. For example, flip a coin twice and let A be the event that the first and second outcome are the same, and let H_1, H_2 be the events that the coin lands "heads" the first and second time, respectively. Then A is independent of H_1 and also of H_2, but it is not independent of the set $\{H_1, H_2\}$.

Suppose that $A_1, \ldots, A_n$ is a set of bad events, and let $p_i = \mathbb{P}(A_i)$. If the events are mutually independent, we have

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = \prod_{i=1}^{n}(1 - p_i).$$

As long as $p_i < 1$ for $1 \leq i \leq n$, we are guaranteed that the probability of no bad events is nonzero. If there is some dependence between the events, then we can use the union bound to obtain

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq 1 - \sum_{i=1}^{n} p_i,$$

but this will only give us a useful bound if the probabilities of bad events are very small. The problem here is that the union bound, in a sense, assumes the "worst possible" dependence: we get the highest possible probability of any bad event if the bad events are disjoint. In most cases, this is far from true. In fact, we often have a situation where there is some dependence between the bad events, but most events A_i are still mutually independent of a large set of other events. We capture this in a *dependency (di)graph* D. Let $V(D) = [n]$, and let the edges be such that A_i is mutually independent of the set $\{A_j \mid (i,j) \notin E(D)\}$. Note that this graph is not unique, not even if we assume that it is edge-minimal. For example, in the coin flip game mentioned earlier, A could have an edge to either H_1 or H_2 in a valid dependency graph. Now we are ready to state the two most commonly versions of Lovász Local Lemma. The first one, the general version, is more general and powerful, but much trickier to work with. The second one, the symmetric version, works well when there is some symmetry among the events, and is easier much easier to apply.

Lemma 2 (Lovász Local Lemma (general)). *Let $A_1, \ldots, A_n$ be events and let D be an associated dependency graph. If there exist a set of real numbers $x_1, \ldots, x_n \in [0, 1)$ such that*

$$\mathbb{P}(A_i) \leq x_i \prod_{(i,j) \in E(D)} (1 - x_j),$$

then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i) > 0.$$

Lemma 3 (Lovász Local Lemma (symmetric)). *Let $A_1, A_2, \ldots, A_n$ be a set of bad events and D an associated dependency graph. If $\mathbb{P}(A_i) \leq p$ and $d_D(A_i) \leq d$ for $1 \leq i \leq n$, and if*

$$ep(d + 1) \leq 1,$$

then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0.$$

Hypergraph proper 2-coloring

Let H be a k-uniform hypergraph. At the beginning of this course, we used the first moment method to show that if H has at most 2^{k-1} edges, then it admits a proper 2-coloring of its vertices, i.e. a coloring with no monochromatic edges. Now, we will strengthen this result to one that does not depend on the number of edges of H, but rather on how much overlap is allowed among them.

Exercise 1. *Use the symmetric version of Lovász Local Lemma to show that if every edge in H shares a vertex with at most $\frac{1}{e}2^{k-1} - 1$ other edges, then H has a proper 2-coloring.* 3

Ramsey numbers

Now, we will look at the Ramsey number $R(3, s)$. This is the smallest number n such that any red/blue coloring of the edges of K_n gives rise to a red copy of K_3 or a blue copy of K_s.

Exercise 2. *Show that if there exist real numbers $p, x, y \in [0, 1)$ such that* 3

$$p^3 \leq x(1 - x)^{3n}(1 - y)^{\binom{n}{s}}$$

and

$$(1 - p)^k \leq y(1 - x)^{\binom{s}{2}n}(1 - y)^{\binom{n}{s}},$$

then $R(3, s) > n$.

Exercise 3. *Find an explicit lower bound for $R(3, s)$.* 3

Proof of the general LLL

Proof. The idea is to show that conditioning on a set of events $\overline{A_j}$ for $j \in S$ does not increase the probability of an event A_i for $i \notin S$ too much. More precisely, we want to show that

$$\mathbb{P}\left(A_i \mid \bigcap_{j\in S} \overline{A_j}\right) \leq x_i,$$

for $S \subseteq [n]$ and $i \notin S$. We will use induction on $|S|$. When $|S| = 0$, we have

$$\mathbb{P}(A_i) \leq x_i \prod_{(i,j)\in E(D)} (1 - x_j) \leq x_i.$$

Now, we split S into two sets: $X = S \cap N_D(i)$ and $Y = S \setminus X$. This gives us

$$\mathbb{P}\left(A_i \mid \bigcap_{j\in S} \overline{A_j}\right) = \frac{\mathbb{P}\left(A_i \bigcap_{j\in X} \overline{A_j} \mid \bigcap_{j\in Y} \overline{A_j}\right)}{\mathbb{P}\left(\bigcap_{j\in X} \overline{A_j} \mid \bigcap_{j\in Y} \overline{A_j}\right)}.$$

For the numerator, we have that

$$\mathbb{P}\left(A_i \bigcap_{j\in X} \overline{A_j} \mid \bigcap_{j\in Y} \overline{A_j}\right) \leq \mathbb{P}\left(A_i \mid \bigcap_{j\in Y} \overline{A_j}\right) \leq \mathbb{P}(A_i) \leq x_i \prod_{(i,j)\in E(D)} (1 - x_j). \tag{1}$$

Let $X = j_1, j_2, \ldots$.For the denominator, we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{j\in X} \overline{A_j} \mid \bigcap_{j\in Y} \overline{A_j}\right) &= \mathbb{P}\left(\overline{A_{j_1}} \mid \bigcap_{j\in Y} \overline{A_j}\right) \mathbb{P}\left(\overline{A_{j_2}} \mid \overline{A_{j_1}} \bigcap_{j\in Y} \overline{A_j}\right) \ldots \\ &\geq \prod_{j\in X} (1 - x_j),\end{aligned}$$

by the inductive hypothesis. This completes the proof that

$$\mathbb{P}\left(A_i \mid \bigcap_{j\in S} \overline{A_j}\right) \leq x_i.$$

Finally, we observe that

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = \mathbb{P}\left(\overline{A_1}\right) \mathbb{P}\left(\overline{A_2} \mid \overline{A_1}\right) \mathbb{P}\left(\overline{A_3} \mid \overline{A_1} \cap \overline{A_2}\right) \ldots \mathbb{P}\left(\overline{A_n} \mid \cap_{i=1}^{n-1} \overline{A_i}\right) \geq \prod_{i=1}^{n} (1 - x_i).$$

□

The following Corollary is sometimes an easy way to apply the general LLL.

Corollary 4. *If* $\mathbb{P}(A_i) < 1/2$ *and* $\sum_{j\in N_D(i)} \mathbb{P}(A_j) \leq 1/4$*, for all* i*, then*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0.$$

Exercise 4. *Prove Corollary 4.* 3

Lopsided LLL

Notice that in the proof above, the only time that we needed the dependency digraph, was when we required

$$\mathbb{P}\left(A_i \mid \bigcap_{j\in Y} \overline{A_j}\right) \leq \mathbb{P}(A_i),$$

in equation 1. Note that this is weaker than what we initially used to define the dependency graph, since we asked for strict equality then. This immediately implies a stronger version of the local lemma, which we obtain by weakening the constraint on the dependency graph. The idea behind this lopsided version is that at first, we wanted to rule out dependence between the bad events, since dependence might decrease the probability of having no bad events compared to when they are independent. However, dependence does not always imply this. The positive correlation implied by the inequality above actually increases the probability of no bad events compared to the independent case, and can therefore be treated as independence for the sake of the lemma.

As an example of an application of the lopsided version of this Lemma, we will consider the probability that a random permutation (chosen uniformly from all permutations on n elements), is a *derangement*: a permutation without fixed points. We'll write this in the language of graph theory. Let $K_{n,n}$ be a complete bipartite graph on partite sets $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$. Let M be a perfect matching sampled uniformly at random from all perfect matchings. Note that such perfect matchings are in bijection with the permutations on $[n]$. For any edge v_iw_i, we let A_i be the event that it is included in M. It is not so hard to see that $\mathbb{P}(A_i) = 1/n$. If the events were independent (which they are not), this would give us $\mathbb{P}(\cap_{i=1}^n \overline{A_i}) = (1 - 1/n)^n = 1/e + o(1)$. The (lopsided) LLL will give us this lower bound, and as it turns out, this is the true probability. We will need the following Theorem:

Theorem 5. *Let $M_0, M_1, \ldots, M_k$ be (not necessarily perfect) matchings in $K_{n,n}$, such that no edge in M_0 shares a vertex with an edge in any of $M_1, \ldots, M_k$. Let B_i be the event that $M_i \subseteq M$. Then, we have*

$$\mathbb{P}\left(A_0 \mid \bigcap_{i=1}^{k} \overline{B_i}\right) \leq \mathbb{P}(A_0).$$

Exercise 5. *Show that Theorem 5 implies the lower bound of $1/e + o(1)$ on the probability* 2
that M is a derangement.

Exercise 6. *Prove Theorem 5. (Not very easy. Combinatorics grad students should try.)* 5