Bond percolation in the grid

This lecture is based on lecture notes by Bartlomiej Blaszczyszyn. Consider a d-dimensional grid: the vertices are the integer points in d-dimensional Euclidean space, with an edge between two vertices if and only if they are at distance 1 from each other. Let $0 \leq p \leq 1$ be fixed. Now, each edge is open with probability p, independently of other edges, and closed otherwise. We let C be the component that contains the origin, i.e. the component on all vertices that can be reached from the origin via open edges. We are interested in the size of this component; in particular, whether it is infinite or not. We let

$$\eta(p) = \mathbb{P}(|C| < \infty).$$

Note that $\eta(p)$ is monotone decreasing with p, and we have $\eta(0) = 1$ and $\eta(1) = 0$. Therefore, there should exist a threshold function p^*, such that

$$\eta(p) = \begin{cases} 1, & \text{if } p < p^*, \\ < 1, & \text{otherwise.} \end{cases}$$

Exercise 1. *Find p^* when $d = 1$.* 1

Proposition 1. *For $d \geq 2$, we have $0 < p^* < 1$.*

Proof. First, we show that $p^* < 1$.

Exercise 2. *Show that p^* is a decreasing function of d. Therefore, it suffices to consider $d = 2$ for this first part.* 2

Exercise 3. *Consider the graph dual of the grid (which looks like a translation of the grid by $(1/2, 1/2)$) where edges in the dual are closed if their dual edge is open and vice versa. Show that $|C| < \infty$ is equivalent to the existence of a cycle (using open edges) in the dual around the origin.* 1

Exercise 4. *Show that* 1

$$\begin{aligned} \mathbb{P}(|C| < \infty) &\leq \mathbb{P}(\textit{there exists a cycle in the dual around the origin}) \\ &\leq \mathbb{E}(\textit{number of cycles in the dual around the origin}) \\ &\leq \sum_{k=1}^{\infty} c(k)(1-p)^k, \end{aligned}$$

where $c(k)$ is the number of such possible cycles of length k.

Exercise 5. *Find an upper bound on $c(k)$ and finish the argument.* 2

To show that $0 < p^*$, we do need to consider all dimensions $d \geq 2$. This time, we will consider paths that start at the origin. If $|C| = \infty$, then there must be infinitely long paths from the origin.

Exercise 6. *Show that* 2

$$\begin{aligned} \mathbb{P}(|C| = \infty) &\leq \mathbb{P}(\textit{there exists a path of length } k \textit{ starting at the origin}) \\ &\leq \mathbb{E}(\textit{number of paths of length } k \textit{ starting at the origin}), \end{aligned}$$

and finish the proof.

$\square$

Exercise 7. *Perform simulations to find estimates for p^* for a chosen set of dimensions. If you want, you can repeat this for "site percolation", where the vertices are open/closed instead of the edges.* 5

Number of infinite components

For any given p, how many infinite components might we see in the bond percolation of the d-dimensional grid? Perhaps surprisingly, this number is either 0 or 1: it is impossible to see multiple infinite components. Note that one can easily construct examples of open/closed configurations of edges that give arbitrarily many infinite components, but such configurations occur with probability 0, as we will see.

First, we need Kolmogorov's zero-one law. For an infinite independent sequence of random variables, a tail event is an event that is independent of any finite subset of those random variables. Kolmogorov's law states that tail events occur with probability zero or one.

Proposition 2. *For a given value of p, the number of infinite components in the bond percolation of the grid is a constant.*

Proof. Let A_k be the event that there are k infinite components, for $k \in \{0, 1, 2, \ldots, \infty\}$. It is easy to see that this is a tail event, and therefore $\mathbb{P}(A_k) \in \{0, 1\}$ for all k. Since the union of these events is the sample space, we must have that exactly one of them has probability 1, and the rest probability 0. $\square$

Next, we need to rule out $2, 3, \ldots, \infty$. Here, we will rule out any finite number greater than 1, using a proof due to Newman and Schulman ("Number and density of percolating clusters". J. Phys. A, 14, (1981), 1735–1743.). The ∞ case is a bit harder, and we won't prove it here, but it can also be ruled out.

Lemma 3. *We have $\mathbb{P}(A_k) = 0$ for $k \in \{2, 3, \ldots\}$.*

Proof. The proof is the same for any $k \in \{2, 3, \ldots\}$, so let's do $k = 3$. We'll also work in $d = 2$, but the proof can easily be generalized. For the sake of contradiction, suppose that $\mathbb{P}(A_3) = 1$. Consider the box $[-m, m]^2$, and let B_m be the event that all three infinite clusters intersect with the box $[-m, m]^2$. We have that $\lim_{m\to\infty} B_m = \cup_{m=1}^{\infty} B_m = A_3$, and therefore $\mathbb{P}(B_m) \to 1$ as $m \to \infty$. The only conclusion that we need here is that there exists some M such that $\mathbb{P}(B_M) > 0$.
Now let $\tilde{B}_m$ be the event that all three infinite clusters touch the boundary of $[-m, m]^2$. Then $B_m \subseteq \tilde{B}_m$ and therefore $\tilde{B}_M > 1$. We can determine whether $\tilde{B}_M$ is true by generating all edges **outside** of the box $[-M, M]^2$. Now, let B be the event that all edges inside $[-M, M]^2$ are open. Since M is finite, we have $\mathbb{P}(B) > 0$. Note that we also have that B and $\tilde{B}_M$ are independent. However, $B \cap \tilde{B}_M = A_1$, and now we have

$$\mathbb{P}(A_1) = \mathbb{P}(B \cap \tilde{B}_M) = \mathbb{P}(B)\mathbb{P}(\tilde{B}_M) > 0,$$

a contradiction. $\square$