Discrete Dynamical Systems

We won't spend a lot of time in this class talking about dynamical systems and Markov Processes since they are mostly outside of the scope, but we will use it as a tool for looking at a simple random walk on a graph.
A (linear) *discrete dynamical system* takes the form

$$\vec{x}(t) = A\vec{x}(t-1), \text{ with some initial condition } \vec{x}(0) = \vec{x}_0.$$

We can write this as a direct formula

$$\vec{x}(t) = A^t\vec{x}_0.$$

We define a *Markov chain* on (finitely many) states $S = \{1, 2, \dots, n\}$ as a process that starts in some state and moves to another state (or stays put) at each discrete time step. So, at each time step t, we have a random variable X_t which represents the state of the process at time t. A *Markov process* is time-homogeneous and memoryless. This means that the random variables $X_1, X_2, \dots$ are identically distributed, and that at each time step, the probability distribution of the next step of the system depends only on the current state, i.e.

$$\mathbb{P}(X_{t+1} = k | X_1 = x_1, \dots, X_t = x_t) = \mathbb{P}(X_{t+1} = k | X_t = x_t).$$

We call these the transition probabilities:

$$p_{ji} = \mathbb{P}(X_{t+1} = j | X_t = i),$$

which we can represent in a $n \times n$ *transition matrix.*

Let an i, j-path be a path $i - i_1 - i_2 - \dots - j$, such that $p_{i_1 i}, p_{i_2 i_1} \dots > 0$. We say that i and j *communicate* if there exist both an i, j-path and a j, i-path. Since communicating is an equivalence relation on the states, this gives us a partition into classes. If all states are in one class, we call the Markov chain *irreducible* or *ergodic*. We say that a transition matrix is *positive* if all its entries are positive (*i.e.* not 0). We say that a transition matrix is *regular* if the matrix M^t is positive for some integer t.

Exercise 1. *Show that an ergodic Markov Process does not imply that the transition matrix is regular.* 1

We call $\vec{x}$ a *distribution vector* if it has nonnegative elements that add up to 1. We then have the following result.

Exercise 2. *Show if A is a transition matrix and $\vec{x}$ is a distribution vector, then $A\vec{x}$ is a distribution vector. In fact, the transformation A always preserves the sum of elements of a vector.* 1

Exercise 3. *Show that if $\vec{x}_t$ represents the distribution of probabilities of X_t taking values in $\{1, 2, \dots, n\}$, then $\vec{x}_{t+1} = A\vec{x}_t$ represents the distribution of probabilities of X_{t+1} taking values in $\{1, 2, \dots, n\}$.* 2

Exercise 4. *Show that an ergodic Markov Process does not imply that the transition matrix is regular.* 2

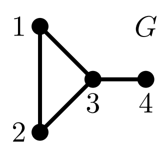
We will use the following Theorem (without proof).

Theorem 1. *If A is a regular $n \times n$ transition matrix, then A has exactly one distribution eigenvector $\vec{x}$ with eigenvalue 1, meaning that $A\vec{x} = \vec{x}$. This is called the equilibrium distribution of A, and denoted by $\vec{x}_{equ}$.*
For any starting distribution $\vec{x}_0$, we have

$$\lim_{t \to \infty} A^t \vec{x}_0 = \vec{x}_{equ}.$$

Number of returns in a random walk on a finite graph

The following example and proof are taken from Bollobás' Modern Graph Theory. We've spent some time thinking about Markov chains on a finite set of states, which can be thought of as random walks on a finite graph. Now we'll look in more detail at the very basic case where a walker is on a finite, simple, undirected graph $G(V, E)$. The walker starts at a starting vertex $v_0 \in V(G)$, and at each time step, they move from their current vertex v to a neighboring vertex of v, choosing one uniformly from the set $\Gamma(v)$ (the neighborhood of v). For example, the following graph G has the transition matrix M:



$$M = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 \end{pmatrix}$$

Exercise 5. *Show that if G is a connected and non-bipartite graph, then its associated transition matrix A is regular.* 4

Exercise 6. *Show that when G is a connected and non-bipartite graph, then* 2

$$\vec{x} = \frac{1}{2m} \begin{pmatrix} d(1) \\ d(2) \\ \vdots \\ d(n) \end{pmatrix},$$

where $m = |E(G)|$ is the equilibrium vector of the random walk on G.

Exercise 7. *Let $\vec{x}_{equi}$ be the distribution of X_t. Instead of considering the next vertex visited, let p_e be the probability that e is the next edge traversed, for any $e \in E(G)$. Find this distribution of probabilities over the edge set of G.* 2

Exercise 8. *Let X be a random variable and a a constant. Show that* $\mathrm{var}(aX) = a^2 \mathrm{var}(X)$. 1

Exercise 9. *Let G be an undirected non-bipartite graph with $v_0 \in V(G)$ and let S_k indicate the number of returns to v_0 of a random walk that starts at v_0 and runs for k steps. Show that the random variable S_k/k concentrates around its mean, i.e. that its $\mathrm{var}(S_k/k) \to 0$ as $k \to \infty$.* 4