Extremal graph theory

Extremal graph theory asks questions of the form: how dense does a graph need to be so that it is guaranteed to have a certain property? The most common property of interest is the appearance of a substructure (subgraph or minor).
We denote by $\mathrm{ex}(n, F)$ the largest number of edges in a graph on n vertices which does not contain F as a subgraph. We will start with a few warm-up exercises.

Exercise 1. *Find $ex(n, P_3)$, where P_3 is the path on 3 vertices.* 2 points

Exercise 2. *Find $ex(n, P_4)$, where P_4 is the path on 4 vertices.* 2 points

Exercise 3. *Find $ex(n, K_{1,t})$.* 2 points

Exercise 4. *Find $ex(n, 2K_2)$.* 2 points

As a warmup example in class, we will prove Mantel's Theorem, which was the precursor of the famous Turán's Theorem. Although it is called Mantel's Theorem, the proof is due to Wijthoff. (Mantel wrote the problem for a publication of the Dutch Royal Math Society, and Wijthoff's submitted solution was chosen for publication.)

Theorem 1. *We have*

$$ex(n, K_3) = \left\lceil \frac{n}{2} \right\rceil \times \left\lfloor \frac{n}{2} \right\rfloor,$$

and the unique graph that achieves this maximum is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

Later, Turán generalized this result to $\mathrm{ex}(n, K_t)$, by showing that the unique maximum graph is the complete $(t-1)$-partite graph with partite sets as balanced as possible. We call this the Turán graph T_{t-1}.

Theorem 2 (Turán)**.** *We have* Thm 7.1.1 p.175

$$ex(n, K_t) = ||T_{t-1}||,$$

and T_{t-1} is the only graph that achieves this maximum number of edges. This implies that

$$\frac{ex(n, K_t)}{\binom{n}{2}} \to \frac{t-2}{t-1}, \quad \text{as } n \to \infty.$$

Exercise 5. *Find $ex(n, G)$, where G is a K_3 with an extra leaf attached. (Or, the graph obtained from K_4 by deleting two incident edges.)* 2 points

The previous exercise is a small hint towards the Erdős-Stone theorem, which says that asymptotically, the chromatic number of a graph determines its extremal numbers and graphs. This theorem resolves the extremal numbers asymptotically for all graphs except bipartite graphs.

Theorem 3 (Erdős-Stone)**.** *Let H be a graph with chromatic number $\chi(H) = t$. We have* Thm 7.1.2 p.178

$$\frac{ex(n, H)}{\binom{n}{2}} \to \frac{t-2}{t-1}, \quad \text{as } n \to \infty.$$