Edge colorings

Just as with vertices, we can ask about proper colorings of edges (or in other words, partitions of the edge set of a graph into matchings). We say that a *proper k-edge-coloring* of a graph G is a function $c : E(G) \to \{1, \ldots, k\}$ such that $e \cap f \neq \emptyset$ implies that $c(e) \neq c(f)$. We let $\chi'(G)$ denote the *edge-chromatic number* of G, which is the smallest value of k such that a proper k-edge coloring is possible. Just as with the vertex colorings, we find two lower bounds based on largest clusters of edges (edges incident to a single vertex) and largest independent sets of edges (matchings). Let $\alpha'(G)$ denote the size of a largest matching in G. We have that

- $\Delta(G) \leq \chi'(G)$
- $\frac{|E(G)|}{\alpha'(G)} \leq \chi'(G)$.

Exercise 1. *Suppose that we try a greedy algorithm similar to the way we started with vertex colorings. Order the edges $e_1, \ldots, e_m$, and color each edge in turn with the lowest available color (lowest color not already used by an edge incident to it). What upper bound on $\chi'(G)$ do we obtain from this approach?* 2 points

As it turns out, edge colorings behave quite differently from vertex colorings. The "easy" lower bound $\Delta(G) \leq \chi'(G)$ is very close:

Theorem 1 (Vizing's Theorem)**.** *We have that* Thm 5.3.2 p.128

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Just because the edge chromatic number is known to be one of two values, it is still difficult to compute exactly. In other words, the class of graphs for which $\chi'(G) = \Delta(G)$ is not easily characterized. We do know that it contains all bipartite graphs.

Proposition 2 (König)**.** *We have that, if G is bipartite,* Thm 5.3.1 p.127

$$\chi'(G) = \Delta(G).$$

We talked about the proof for the latter proposition in class. Try to read this proof and understand it well. Then, you will notice why it fails on general graphs. The proof for Vizing's Theorem uses a couple of tricks to get around this issue. The tricks should be reminiscent of the proof of the 5-color Theorem for vertex-coloring planar graphs.

Exercise 2. *Without using Proposition 5.3.1, show that $\chi'(G) = k$ for every k-regular bipartite graph G.* 3 points

Exercise 3. *Suppose that G is a 3-regular graph on an even number of vertices, and that G has a Hamilton cycle. (A cycle in G that uses all vertices.) Show that $\chi'(G) = \Delta(G) = 3$.* 2 points

Exercise 4. *An $n \times n$ matrix with entries from $\{1, \ldots, n\}$ is called a* Latin square *if every element of $\{1, \ldots, n\}$ appears exactly once in each column and once in each row. Rephrase the problem of finding Latin squares as a graph coloring problem.* 3 points