Planar graphs

So far, we have thought about graphs only as abstract objects. They may be drawn on a piece of paper as a way of better understanding them, but the drawing is not inherently part of the object. Now, we will consider their drawings (in the plane) more formally. We say that an *embedding* of a graph G in the plane assigns to each vertex a point and to each edge a line that connects two points. Some topology can go in to defining what a line is exactly, but we will stick to the intuitive definition here. We call such an embedding *planar* if no two edges cross, and we call a graph with such an embedding a *plane graph*. If a graph has a planar embedding, we call it a *planar graph*. Note the difference: a plane graph is not the same object as the standard graphs we have talked about so far; it can be thought of as a point set in the plane. A planar graph is just a graph as usual that happens to have a specific property.

Forbidden substructures

We consider three types of substructures of a graph G:

- Induced subgraph: a graph that may be obtained from G by vertex deletion (which deletes incident edges as well),
- Subgraph: a graph that may be obtained from G by vertex deletion and/or edge deletion,
- Minor: a graph that may be obtained from G by vertex deletion, edge deletion, and/or edge contraction.

An *edge contraction* of an edge $uv \in E(G)$ means deleting the edge, and then merging its two vertices. The neighborhood of the new, merged vertex is the union of the neighborhoods of u and v. It is not so hard to see that the class of planar graphs is closed under vertex deletion and edge deletion. It is also closed under edge contraction. (Convince yourself of this.) In fact, the class of planar graphs is exactly the class of graphs that are free of a forbidden set of minors:

Theorem 1 (Wagner's Theorem). *A graph G is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.* Thm 4.4.6 p.107

Faces

Once we have embedded a planar graph in the plane, we see that the plane is divided into connected regions. Again, we will stick with the intuitive definition here, without getting into a topological definition of connectedness. If and only if a graph has a cycle, we have at least two of these regions. (Convince yourself of this.) We call these the *faces* of the plane graph. Even though a planar graph may have multiple ways of being embedded in the plane, it turns out that the number of faces is determined by the number of edges and vertices.

Theorem 2 (Euler's Formula). *A graph connected plane graph G with n vertices, m edges and f faces satisfies* Thm 4.2.9 p.97

$$n - m + f = 2.$$

The proof is written out in the book, but intuitively: we can check that this formula holds for trees (minimally connected), since we know that trees on n vertices have $n-1$ edges, and

only 1 face when embedded in the plane. Now, every new edge lies on a cycle, and therefore it cuts one face into two. Therefore, $n - m + f$ stays constant.
Before you start the next exercise, convince yourself that being embeddable in the plane or on a sphere are equivalent.

Exercise 1. *A soccer ball is made up of hexagons and pentagons. If we think of those as the faces of a graph, it creates a 3-regular graph. Use Euler's formula to determine the possible number of hexagons and pentagons on a soccer ball.* 3 points

Exercise 2. *Show that for any planar graph G on n vertices and m edges, we have* 3 points

$$m \leq 3n - 6.$$

Exercise 3. *Show that for any planar graph G we have $\delta(G) \leq 5$.* 3 points

Chromatic number

A k-(vertex)-coloring of a graph G is a function $c : V(G) \to [k]$ such that $uv \in E(G)$ implies $c(u) \neq c(v)$. Then we let the *chromatic number*, denoted $\chi(G)$, be the smallest value of k for which G admits a k-coloring. Sometimes this type of coloring is referred to as a *proper coloring*. Note that the 2-colorable graphs are exactly the bipartite graphs. We can think of a k-coloring as a partition $V = V_1 \cup \cdots \cup V_k$ so that every V_i is an independent set of vertices.
Let $\omega(G)$ be the *clique number* of G, which denotes the cardinality of a largest set of vertices $S \in V(G)$ such that $uv \in E(G)$ for every $u, v \in S$. Similarly, let $\alpha(G)$ be the *independence number* of G, which denotes the cardinality of a largest set of vertices $S \in V(G)$ such that $uv \notin E(G)$ for every $u, v \in S$.
As an exercise, show the following two inequalities, which hold for any graph G:

- $\chi(G) \geq \omega(G)$
- $\chi(G) \geq \frac{n}{\alpha(G)}$.

For each of the two inequalities above, as an exercise, find an example of a graph that shows that they are not tight.
To find an upper bound on the chromatic number, we start with an easy, greedy algorithm. Suppose that the vertices are ordered: $v_1, \ldots, v_n$. Color each v_i with the "lowest" available color. Here, a color is available if it is not yet in use by any of the neighbors of v_i. Therefore, we color vertex v_1 with color 1, vertex v_2 with color 1 if $v_1v_2 \notin E(G)$ and with color 2 if $v_1v_2 \in E(G)$, etc. Since a vertex has at most $\Delta(G)$ neighbors, there is always a color available in the set $\{1, \ldots, \Delta(G) + 1$. This gives us the upper bound:

$$\chi(G) \leq \Delta(G) + 1.$$

Exercise 4. *Show that the greedy algorithm can be arbitrarily bad, by finding, for example, a bipartite graph and an ordering on its vertices so that the greedy algorithm uses K colors, where K is any large number.* 3 points

Exercise 5. *Here is another idea for a greedy algorithm: find a maximum independent set S_1 in G, and give it color 1. Then find a maximum independent set S_2 in $G - S_1$ ans give it color 2, etc. Can this algorithm do badly as well? (e.g. are there bipartite graphs for which it uses an arbitrarilty large number of colors?)* 5 points

Coloring planar graphs

One of the most famous theorems in graph theory is the 4-color Theorem.

Theorem 3 (Four Color Theorem)**.** *Every planar graph is 4-colorable.* Thm 5.1.1 p.120

Despite being easy to state, it is incredibly difficult to prove. There exists a proof that is heavily aided by exhaustive computer search, and finding an “elegant” proof remains one of the holy grails of combinatorics. In class, we discussed the much more easily proved 5-color theorem.

Theorem 4 (Five Color Theorem)**.** *Every planar graph is 5-colorable.* Thm 5.1.2 p.120

Exercise 6. *Try to adapt the 5-color theorem proof from the book to prove the 4-color theorem instead. This (most likely) fails. Where does it go wrong?* 3 points