Block decomposition

We have talked about connected components of graphs. If a graph is not connected, then it has multiple such components. Intuitively, it is easy to understand what those are. Here is a more formal definition: For $G(V, E)$ a graph, a *connected component* of G is a maximal connected subgraph. In other words, if H is a connected component of G, then H is a connected subgraph if G and there is no other connected subgraph H' such that $H < H'$ (H is not a proper subgraph of any connected subgraph H'). From this definition, we can replace the word connected with 2-connected. Such subgraphs are blocks. However, we would like to be able to fully decompose a graph G into its blocks, just like we did with the connected components. A *decomposition* of a graph G is a set of subgraphs $H_1, \ldots, H_k$ of G and whose union is G, and such that $E(H_1), \ldots, E(H_k)$ is a partition of $E(G)$.
So, we allow a few more substructures to be called blocks. Formally, a *block* is a maximal connected subgraph without a cut-vertex. Now, G is the union of its blocks, and every edge lies in a unique block.

Exercise 1. *Show that the blocks of G are indeed a graph decomposition.* 2 points

Lemmas 3.1.2 and 3.1.3 show us that block decompositions are a natural way to understand cycles and cuts in connected graphs. First, we make an observation about minimal disconnecting edge sets (sets of edges $S \subseteq E(G)$ such that $G - S$ is disconnected). For two subsets of vertices A and B, we let

$$[A, B] := \{ab \in E(G) \mid a \in A,\ b \in B\}.$$

We let $G[A]$ denote the subgraph of G *induced* by A. This is the subgraph of G who vertex set is A and who edge set is all edges of G that have both endpoints in A.

Claim 1. *A minimal disconnecting edge set in a connected graph G is of the form $[A, \overline{A}]$, where $A \subseteq V(G)$ and $\overline{A} = V(G) \setminus A$. Furthermore, $G[A]$ and $G[\overline{A}]$ are both connected.*

Proof. Let S be a minimal disconnecting edge set. Then $G - S$ has multiple connected components. For an arbitrary component G_1 of $G - S$, note that all edges in S that have exactly one endpoint in G_1 form an edge cut. Since S is minimal, we must have that all edges of S have exactly one endpoint in G_1. Therefore, we let $A = V(G_1)$. All that remains to show is that $G[\overline{A}]$ is connected (i.e. that $G - S$ has excatly two connected components). Finish this proof in the next exercise. □

Exercise 2. *Finish the proof of Claim 1.* 2 points

Now, we call an *edge cut* of a graph G a set of edges S that is of the form $[A, \overline{A}]$ for some $A \subseteq V(G)$, and a *bond* is a minimal nonempty edge cut. Note that bonds exist even for disconnected graphs.

Lemma 2. *Let G be any graph.* 3.1.2, p.60

(i) The cycles of G are precisely the cycles of its blocks.

(ii) The bonds of G are precisely the bonds of its blocks.

Lemma 3. *Let G be any graph. The following statements are equivalent for any $e, f \in E(G)$:* 3.1.3, p.61

(i) The edges e, f belong to common block of G.

(ii) The edges e, f belong to common cycle of G.

(iii) The edges e, f belong to common bond of G.

If A is the set of cutvertices of G, and $\mathcal{B}$ the set of blocks, then we have a natural bipartite graph on $A \cup \mathcal{B}$ formed by edges aB if $a \in B$.

Lemma 4. *The block graph of G forms a tree.* 3.1.4, p.61

Exercise 3. *Prove Lemma 3.1.4.* 2 points

Menger's Theorem

We have mentioned the idea of Menger's Theorem a few times, and now we present it in detail. Section 3.3 has two versions of Menger's Theorem. We will discuss both versions and some of their proofs in class.

Theorem 5. *For a graph G with $A, B \subseteq V(G)$ (not necessarily disjoint), the minimum number of vertiecs separating A from B is equal to the maximum number of disjoint $A - B$ paths in G.* 3.3.1, p.67

Theorem 6. *A graph G is k-connected if and only if it contains k internally disjoint paths between any pair of vertices. A graph G is k-edge-connected if and only if it contains k edge-disjoint paths between any pair of vertices.* 3.3.6, p.72

Exercise 4. *Suppose that a graph G is 2-connected. Let P be a vw-path in G for some pair of vertices $v, w \in V(G)$. Must there exist another vw-path P' that is internally disjoint from P?* 2 points

Exercise 5. *Use Menger's Theorem to prove Hall's Theorem. (You only need to prove the sufficiency of Hall's Condition, not the necessity.)* 2 points