This week we will begin with talking a bit about bipartite graphs from Notes 1, which will then lead us into Hall's Theorem. I repeat the two exercises from Notes 1 that pertain to bipartite graphs, and you may do them for either HW1 or HW2.

Exercise 1. *Let $V = \{0,1\}^d$. In other words, V is the set of all binary strings of length d. The d-dimensional hypercube Q_d is the graph on V such that two vertices in V share an edge if and only if the strings differ in exactly one bit. Show that the hypercube Q_d is a bipartite graph, for $d = 1, 2, \ldots$* 1 point

Exercise 2. *Show that if a bipartite graph G is k-regular, meaning that $d(v) = k\ \forall v \in V(G)$, then the partition classes have the same size.* 1 point

Hall's Theorem

Suppose that we have a set of n students and a set of at least n sandwiches, each with different ingredients. Each student lists the sandwiches that they would be happy to eat (based on preference, dietary restrictions, etc..). Is is possible to distribute the sandwiches so that everyone is happy? Suppose that a subset X of the students is interested only in a subset Y of the sandwiches such that $|Y| < |X|$. For example, perhaps there are two students who have only listed the same single sandwich. This is clearly a problem that prevents everyone being happy. As it turns out, this is the only problem we need to worry about.

Theorem 1 (Hall's Theorem.)**.** *Let G be a bipartite graph with partite sets A and B. We have that $|N(S)| \geq |S|$ for every subset $S \subseteq A$ if and only if G contains a matching of A to B.* 2.1.2, p.38

In the above, the notation $N(S)$ indicates the neighborhood of the set S, meaning the set of all vertices in $V(G)$ that have at least one neighbor in S. In class, we will go over the second proof in Diestel for Hall's Theorem first. Then, we will talk about augmenting paths, both for the first proof of Hall's theorem and for König's theorem.

Exercise 3. *Show that if a bipartite graph G is k-regular, then it satisfies Hall's condition.* 1 point

Solution.

Exercise 4. *Let M be a matching in a bipartite graph G. Show that if M is sub-optimal (contains fewer edges than some other matching in G), then G contains an augmenting path with respect to M.* 2 points

Exercise 5. *Does the above result generalize to non-bipartite graphs?* 2 points

Exercise 6. *Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$. Suppose that G has a matching M_1 that covers $X_1 \subseteq V_1$ and a matching M_2 that covers $X_2 \subseteq V_2$. Show that G has a matching that covers $X_1 \cup X_2$.* 2 points

Solution. Consider the graph on $V(G)$ with edge set $M_1 \cup M_2$. This is a subgraph with max degree 2, and is therefore a union of paths and cycles. We will pick a subset $M \subseteq M_1 \cup M_2$ that forms a matching and covers $X_1 \cup X_2$.

- Any cycle formed by $M_1 \cup M_2$ must be an even cycle, since the graph is bipartite and also since M_1 and M_2 must alternate on the cycle. From each cycle, add the edges in M_1 (or M_2) to M. These are sets of disjoint edges that cover all the vertices of the cycles.

- Any odd path formed by $M_1 \cup M_2$ is either an M_1- or an M_2-augmenting path, in which case we pick the edges of the path that are in M_2 or M_1, respectively, and add them to M. These are sets of disjoint edges that cover all the vertices of the odd path.
- Any even path formed by $M_1 \cup M_2$, either has both of its endpoints in V_1 or both in V_2. If both endpoints are in V_1, then pick the edges of M_1 and add them to M. These edges cover all of the vertices on the path except for one of the endpoints. However, this endpoint was never covered by M_1 and is therefore not in X_1. Similarly, if both endpoint of the path are in V_2, pick the edges on the path that are in M_2 and add them to M.

The resulting set of edges M is disjoint, and covers all the vertices that are covered by $M_1 \cup M_2$, except for those that were not in X_1 or X_2. In other words, M covers $X_1 \cup X_2$.

Exercise 7. *Let M be a matching in a simple graph G. Show that G has a maximum matching that covers all of the vertices covered by M. Hint: over all the maximum matchings in G, consider one with the highest number of edges in common with M.* 2 points

Solution. Let M' be a maximum matching with the highest number of edges in common with M. Consider the graph $\{V(G), M \cup M'\}$. As we have seen, this graph has maximum degree 2, and is therefore a disjoint union of cycles and paths. Suppose that there is a vertex v that is covered by M and not covered by M'. Then, v is an endpoint of a path P in the graph $\{V(G), M \cup M'\}$ (since it has degree 1). The path P must have an even number of edges, or else it is an M'-augmenting path, which contradicts M' being a maximum matching. However, if P is even, then we can remove the M'-edges of P from M' and replace them with the M-edges from P, to obtain a new matching that is also maximum and has more edges in common with M than M'. Therefore, such a vertex v cannot exist.
König's Theorem is an important example of a type of min/max result that we will see a couple of times in this course. Minimizing one type of structure is equivalent to maximizing another. This result is closely related to Hall's Theorem, and Menger's Theorem and the Min-cut Max-flow Theorem.

Theorem 2 (König's Theorem.). *If G is a bipartite graph, then the maximum cardinality of a matching is equal to the minimum cardinality of a vertex cover of its edges.* 2.1.1, p.37

Exercise 8. *The proof of König's Theorem in the book is quite dense. Write your own version of the proof with a bit more explanation of some of the details.* 3 points

Exercise 9. *Over all connected graphs on n vertices, what is the lowest value that the size of a minimum vertex cover can take (in terms of n)? What is the highest?* 2 points

For both cases, give an example of a graph with a lowest/highest value, and argue why there are no graphs with lower/higher values.

Solution. When $n = 0$ or $n = 1$, G has no edges, and the cardinality of a minimum vertex cover, denoted $\beta(G)$, is trivially 0. For $n = 2, 3, \ldots$, G must have at least one edge, and therefore $\beta(G) \geq 1$. There exist connected graphs such that a single vertex is incident to all edges (stars), which achieve this. (They are the only graphs that do.)
If we have a graph G with some $\beta(G)$, then adding an edge to G can never decrease β (either the edge is already covered by a minimum cover, or it increases the number of vertices needed to cover all edges). Therefore, the complete graphs K_n must maximize $\beta(G)$. In K_n, and in

any vertex cover Q, there can be no two vertices v, w that are not in Q, since the edge $\{v, w\}$ would then not be covered. It is possible for only one vertex v to not be in Q, since no edge can have both of its endpoints outside of Q. Therefore, $\beta(K_n) = n - 1$ for $n = 1, 2, \ldots$ and $\beta(G) \leq n - 1$ for any graph G.