Miscellaneous

On Tuesday, we talked a bit about graphs and matroids. This is not a formal part of this course, but I recommend this text as a friendly introduction if you'd like to see one:
`http://reu.dimacs.rutgers.edu/~ecatania/Matroids.pdf`
A few exercises for this week:

Exercise 1. *In class, we defined the* dual *graph G' of a planar graph G: the dual graph has a vertex for each face of G, and two vertices have an edge if the respective faces share an edge.* 3 points

Let G be a planar graph and T a spanning tree of G. Show that the dual edges of the complement of T form a spanning tree of the dual graph G'.

Exercise 2. *Let G be a bipartite graph with $\Delta(G) = r$. Show that the edge set of G can be partitioned into r matchings.* 3 points

Exercise 3. *Let $\alpha'(G)$ be the size of a maximum matching, and $\beta'(G)$ the size of a minimum edge cover (a set of edges such that each vertex is incident to at least one edge in the set). Let G be an n-vertex graph without isolated vertices. Show that* 3 points

$$\alpha'(G) + \beta(G) = n.$$

Introduction to the chromatic polynomial

Here, we will follow some of the notes by Andrew Goodall:
`https://iuuk.mff.cuni.cz/~andrew/VKKI.pdf`
The notes here are a bit more condensed.
Let G be a graph. We allow loops and multi-edges because we will be working recursively with deletions and contractions. We would like to find a polynomial $P_G(z)$ of order $n = |G|$, such that for $k \in \mathbb{N}$, $P_G(k)$ is equal to the number of proper k-colorings of G. It is not obvious that such a polynomial exists, but we will show that it does by construction.
For a real number z and natural number i, we define the *falling factorial* by

$$z^{\underline{i}} = (z) \cdot (z-1) \cdot \ldots \cdot (z-i+1).$$

For $1 \leq i \leq n$, let $a_i(G)$ be the number of partitions of $V(G)$ into i classes, such that the classes are independent sets. (Note that classes must be nonempty.) Convince yourself that if $k \in \mathbb{N}$ $k \geq i$, then there are $k^{\underline{i}}$ proper k-colorings of G that use i colors. Also note that if $k < i$, then $k^{\underline{i}} = 0$. We let

$$P_G(z) = \sum_{i=1}^{n} a_i(G) z^{\underline{i}}.$$

Check that $P_G(z)$ is indeed a polynomial in z of order n such that for $k \in \mathbb{N}$, $P_G(k)$ is equal to the number of proper k-colorings of G.

Exercise 4. *Let* 2 points

$$P_G(z) = \sum_{i=1}^{n} a_i(G) z^{\underline{i}} = \sum_{i=1}^{n} c_i z^i.$$

Show that $c_n = 1$ and $c_{n-1} = -m$, where m is the number of edges of G.

Exercise 5. *Find* $P_G(z)$ *when* $G \sim P_n$*, the path graph on* n *vertices.* 2 points

Exercise 6. *Find* $P_G(z)$ *when* $G \sim C_n$*, the cycle graph on* n *vertices.* 2 points

Exercise 7. *Find* $P_G(z)$ *when* $G \sim W_n$*, the wheel graph on* n *vertices.* 2 points