Euler tour

An *Euler tour* (or circuit) in a graph G is a closed walk that visits every edge exactly once. It is not so difficult to see that this implies that all vertex degree are even. Similar to Hall's and Menger's theorems, it turns out that this simple necessary condition is also sufficient. This is widely regarded as the oldest known theorem in graph theory, and is due to Euler.

Theorem 1 (Euler)**.** *A connected graph G has an Euler tour if and only if all vertices of G have even degree.* Thm 1.8.1 p22

Exercise 1. *A complete graph K_n has an Euler tour if n is odd, by Euler's theorem. What is the length of a longest tour (closed walk that does not repeat edges) in K_n when n is even?* 2 points

Exercise 2. *A hypercube Q_n has an Euler tour if n is even, by Euler's theorem. What is the length of a longest tour in Q_n when n is odd?* 3 points

Hamilton cycles

A *Hamilton cycle* in G is a cycle in G of length $n = |G|$. For Hamilton cycles, we do not have such a nice necessary and sufficient condition as we do for Euler tours. In class, we covered the proofs for the following two theorems, which give sufficient (but far from necessary) conditions for a graph to be Hamiltonian (have a H.C.).

Theorem 2 (Dirac)**.** *Suppose $|G| \geq 3$, then G is Hamiltonian if $\delta(G) \geq n/2$.* Thm10.1.1

Theorem 3. *Suppose $|G| \geq 3$, then G is Hamiltonian if $\alpha(G) \leq \kappa(G)$.* Thm10.1.2

Exercise 3. *Show that every hypercube Q_n for $n \geq 2$ is Hamiltonian.* 2 points

Exercise 4. *An oriented complete graph is called a tournament. Show that every tournament contains a dirceted Hamilton path.* 2 points