Ramsey theory

Ramsey theory asks extremal questions from a slightly different perspective from Turan theory. If we fix the number of vertices, then a high density of edges guarantees the appearance of certain subgraphs, while a low densite of edges guarantees subgraphs in the complement of the graph. Instead of thinking of graphs and their complements, we can think of taking the complete graph K_n and coloring the edges red/blue. The questions are now of the form: what is the smallest number of vertices such that every red/blue coloring of the edges of K_n has either a red copy of G or a blue copy of H. In 1930, Ramsey proved that such a number exists.

Theorem 1. *For every $s, t \in \mathbb{N}$, there exists an n such that every red/blue coloring of the edges of K_n has either a red copy of K_s or a blue copy of K_t.* Thm 9.1.1 p.284

We let $R(s,t)$ be the smallest such n, and let $R(s) = R(s,s)$. We did most of the following proofs in class.

Exercise 1. *Write the full proof that $R(3) = 6$. (Lower and upper bound.)* 1 point

Exercise 2. *Show that $R(2,s) = s$.* 1 point

Exercise 3. *Show that $R(3,4) \leq 9$. (We did an example in class to show that $R(3,4) \geq 9$.)* 2 points

Exercise 4. *Show that* 2 points

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Probabilistic method

A *sample space* is a set of elements that we call *outcomes*. An *event* is a subset $A \subseteq \Omega$. The *event space* $\mathcal{F}$ is a sigma-algebra: a collection of subsets $\mathcal{F} \subseteq 2^\Omega$ that includes Ω, is closed under taking complements and countable unions. For our purposes it is fine to assume $\mathcal{F} = 2^\Omega$. A *probability measure* is a function $\mathbb{P} : \mathcal{F} \to [0,1]$, such that $\mathbb{P}(\Omega) = 1$, and $\mathbb{P}$ is countable additive, meaning that $\mathbb{P}(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mathbb{P}(A_i)$, for disjoint events $A_1, A_2, \ldots$. Now, a *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$.

A *random variable* is a measurable function $X : \Omega \to S$, where S is the *state space* of X. We will always deal with $S \subseteq \mathbb{R}$, and most of the time we will deal with discrete nonnegative random variables, such that $S \subseteq \mathbb{N}_0$. We use the notation $\mathbb{P}(X \in B) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in B\})$. (Note that in that expression $B \subseteq \mathbb{R}$.) We let I_A be an *indicator random variable* for the event A. This means that

$$I_A = \begin{cases} 1, & \text{if } A \text{ occurs} \\ 0, & \text{otherwise.} \end{cases}$$

In the case of a discrete random variable, we define the *expectation* of X as

$$\mathbb{E}(X) = \sum_{x \in S} x \mathbb{P}(X = x).$$

We say that two events A_1 and A_2 are *independent* if

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2).$$

In order to understand this idea more intuitively, we need the following definition. We let

$$\mathbb{P}(A_1|A_2) = \frac{\mathbb{P}(A_1 \cap A_2)}{\mathbb{P}(A_2)}.$$

The *conditional probability* $\mathbb{P}(A_1|A_2)$ indicates the probability of event A_1 under the assumption of A_2. Another way of thinking about conditional probabilities is the following: we assume that $\mathbb{P}(A_2) = 1$ and therefore $\mathbb{P}(\omega) = 0$ for any $\omega \notin A_2$. We then obtain a new probability measure by scaling $\mathbb{P}(\omega)$ by $1/\mathbb{P}(A_2)$ for each $\omega \in A_2$ so that our total probability remains 1. Now, we see that two events A_1 and A_2 are independent if and only if

$$\mathbb{P}(A_1|A_2) = \mathbb{P}(A_1).$$

In other words, the events A_1 and A_2 are independent if knowledge of one event does not yield information about the probability of the other.
Similarly, for random variables X, Y, we say that X and Y are *independent* if

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x)(Y = y), \text{ for all } x, y.$$

The following lemma is one of the reasons that first-moment method proofs are usually surprisingly straight-forward. We can ignore dependence when working with the expectation of the sum of sets of random variables.

Lemma 2. *We have* linearity of expectation*:*

$$\mathbb{E}\left(\sum_i X_i\right) = \sum_i \mathbb{E}(X_i),$$

whether the events X_i are dependent or not.

Before we get into more complicated first-moment lemmas, we make a simple observation:

Proposition 3. *We have*

$$\mathbb{P}(X \leq \mathbb{E}(X)) > 0,$$
$$\mathbb{P}(X \geq \mathbb{E}(X)) > 0.$$

This simple observation gives us a powerful method of proving the existence of structures without finding explicit constructions. If $X(G)$ is an invariant of, for example, graphs, and we sample graphs from some distribution, then we immediately know that graphs with $X \geq \mathbb{E}(X)$ as well as graphs with $X \leq \mathbb{E}(X)$ exist.

Lemma 4 (Markov's inequality)**.** *For a non-negative random variable, we have*

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

The following is obtained from Markov by letting $t = 1$.

Lemma 5 (First moment method)**.** *For a non-negative, integer-valued random variable, we have*

$$\mathbb{P}(X > 0) \leq \mathbb{E}(X).$$

Let $G = (V, E)$ be a graph, and let $n = |V|$ be the number of vertices in G. A k-edge-coloring of a graph is a coloring that uses k colors. We wish to find the largest n s.t. there exists a 2-edge coloring such that G contains no monochromatic K_s.

Exercise 5. *Show that $R(s,s) > n$ if $\binom{n}{s}2^{1-\binom{s}{2}} > 1$. (Can you replace the $>$ by $\geq$ in that last inequality?)* 2

Exercise 6. *Show that $R(s,s) > n - \binom{n}{s}2^{1-\binom{s}{2}}$ for any n.* 1