Introduction to graphs

Degrees

Read Sections 1,2,3,5,6 from Chapter 1. These sections contain a lot of definitions that we will not imemediately use, but you will at least remember where to find them when they come up later. A graph is a pair $G(V, E)$, where V is a set of *vertices* (nodes) and $E \subseteq \binom{V}{2}$ a set of *edges* (links). In this course, we will almost always consider graphs to be *simple*, meaning that they have no self-loops or multiple edges, and that edges are undirected. Let $d(v)$ indicate the *degree* of a vertex $v \in V(G)$, i.e. the number of edges that are *indicent* to it (contain v), or the number of vertices *adjacent* to v (that share an edge with v). As a warm-up exercise, we prove the Handshake Lemma.

Lemma 1 (Handshake Lemma.). *For any graph G, we have* p.5

$$2|E| = \sum_{v \in V} d(v).$$

It follows immediately that the number of odd-degree vertices in a graph is always even.

Exercise 1. *Is it possible to have a graph G (on at least 2 vertices) such that $d(v) \neq d(u)$ for all $v \neq u \in V(G)$?* 1 point

We denote the average degree of G as $d(G)$, the minimum as $\delta(G)$ and the maximum as $\Delta(G)$. A *subgraph* of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Note that we cannot take any subset of $V(G)$ and $E(G)$, as this is not necessarily a graph. Of course a graph can have a large average degree and small minimum degree. However, if a graph has high average degree it must at least have a subgraph with a high minimum degree. We show the following Proposition.

Proposition 2. *Every graph G with at least one edge has a subgraph H with $\delta(H) \geq d(G)/2$.* 1.2.2, p.6

Paths and cycles

A *path* is a graph of the form $P(V, E)$ with $V = \{v_0, \ldots, v_k\}$ and $E = \{v_0v_1, \ldots, v_{k-1}v_k\}$. A *cycle* is a graph of the form $C(V, E)$ with $V = \{v_0, \ldots, v_k\}$ and $E = \{v_0v_1, \ldots, v_{k-1}v_k, v_1v_k\}$. We say that a graph is *connected* if there exists a path in the graph between any pair of vertices. In class, we will prove the following Proposition.

Proposition 3. *Every graph G contains a path of length at least $\delta(G)$ and a cycle of length at least $\delta(G) + 1$.* 1.3.1, p.8

A *walk* in a graph is a generalization of a path: it may repeat vertices and edges. A *closed walk* in a graph is a generalization of a cycle: it may repeat vertices and edges.

Exercise 2. *Show that if a simple graph G contains an odd closed walk, it contains an odd cycle.* 2 points

A *trail* in a graph is a generalization of a path: it may repeat vertices, but not edges. A *tour* in a graph is a generalization of a cycle: it may repeat vertices, but not edges. An *Euler tour* in a graph is a tour that visits every edge. One of the earliest and most famous theorems in graph theory is due to Euler. It is inspired by the Seven Bridges of Königsberg problem.

Proposition 4. *A connected graph has an Euler tour if and only if every vertex has even degree.* 1.8.1, p.22

Trees

A *tree* is an acyclic connected graph. The following Theorem summarizes a set of equivalent definitions of trees, which are each useful in different contexts.

Theorem 5. *The following are equivalent for a graph T:* 1.5.1, p.14

(i) T is a tree;

(ii) any two vertices are linked by a unique *path in T;*

(iii) T is minimally connected ($T - e$ is disconnected for every $e \in E(T)$);

(iv) T is maximally acyclic ($T + e$ has a cycle for every $e \in \overline{E(T)}$).

Exercise 3. *Prove Theorem 5.* 3 points

Exercise 4. *Prove that every tree has at least $\Delta(T)$ vertices of degree 1 (leaves), where Δ indicates the maximum degree of T.* 1 points

Exercise 5. *Show that a tree without a vertex of degree 2 has more leaves than inner vertices. Can you find a very short proof of this?* 1 points

Bipartite graphs

A *bipartite graph* is a graph $G(V, E)$ if there exists a bipartition $V = V_1 \cup V_2$ (with $V_1 \cap V_2 = \emptyset$) such that every edge has one endpoint in V_1 and one endpoint in V_2. We discuss the following equivalent definition in class.

Proposition 6. *A graph is bipartite if and only if it contains no odd cycles.* 1.6.1, p.18

Exercise 6. *Let $V = \{0, 1\}^d$. In other words, V is the set of all binary strings of length d. The d-dimensional hypercube Q_d is the graph on V such that two vertices in V share an edge if and only if the strings differ in exactly one bit. Show that the hypercube Q_d is a bipartite graph, for $d = 1, 2, \ldots$* 1 point

Exercise 7. *Show that if a bipartite graph G is k-regular, meaning that $d(v) = k\ \forall v \in V(G)$, then the partition classes have the same size.* 1 point