Groups acting on themselves by left multiplication

In earlier lectures we already touched on a fundamental and perhaps surprising concept: the symmetric groups contain all possible groups. More precisely

Theorem (Cayley's Theorem)**.** *Every group is isomorphic to a subgroup of some symmetric group. If G is a group of order n, then G is isomorphic to a subgroup of S_n.* Cor. 4 p.120

We proved this by letting G act on itself by left multiplication and noticing that this action is faithful. Then, we obtain an injective permutation homomorphism of G into S_n. Now we generalize this by cnsidering the left multiplication action of G on the set of cosets of a subgroup $H \leq G$.

Theorem. *Let G be a group, H any subgroup of G and let G act on the set A of left cosets of H by left multiplication. Let π_H be the associated permutation representation. Then* Thm. 3 p.121

(1) G acts transitively on A,

(2) the stabilizer G_H is the subgroup H itself,

(3) the kernel of π_H is $\cap_{g \in G} gHg^{-1}$, and this is the largest normal subgroup contained in H.

Finally, we obtain a nice little corollary that generalizes the result that subgroups of index 2 are normal in G. We will walk through the proof together in class.

Corollary. *If G is a finite group of order n and p is the smallest prime dividing n, then any subgroup of index p is normal.* Cor. 5 p.120

Conjugacy Classes

We consider the action of G on itself via conjugation. Let G be a group and let the action be given by

$$g \cdot h = ghg^{-1}, \quad g, h \in G.$$

Then the orbit of an element $a \in G$ is given by

$$O_h = \{a \in G \mid a = ghg^{-1}\}.$$

And the stabilizer is given by

$$G_h = \{g \in G \mid ghg^{-1} = h\} = C_G(h).$$

Therefore, by the previous section, we have

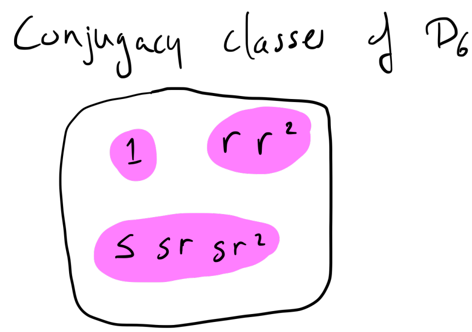
$$|O_h| = |G : C_G(h)|.$$

We have seen that the orbits partition the group into classes, which we will call the *conjugacy classes*. The classes of size 1 are exactly the elements that commute with everything, i.e. $Z(G)$. We can now count the elements of G by adding up the cardinalities of the classes, which is summarized in the Class Equation.

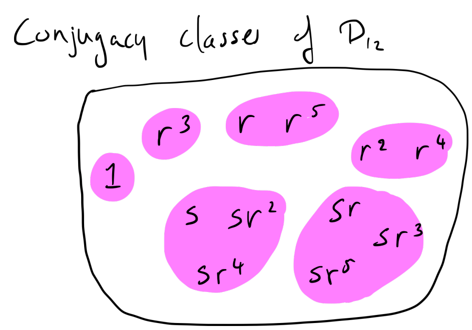
Theorem (Class Equation.)**.** *Let G be a finite group and let $g_1, \ldots, g_r$ be representatives of the distinct conjugacy classes not contained in $Z(G)$. Then* Thm. 7 p.124

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)|.$$

We can think of the conjugacy classes of an element as elements that are "the same from a different perspective", similarly to a change of basis in linear algebra. For example, in D_6, the elements r and r^2 are conjugate via a reflection (rotation 120° clockwise is the same as a rotation −120° if we reflect). Similarly, the reflections s, sr, sr^2 are all similar and conjugate via a rotation.



Below are the conjugacy classes for D_{12}. Can you visualize these?



The symmetric group S_n has a nice description of its conjugacy classes.

Proposition. *Let $\sigma, \tau \in S_n$ and suppose σ has cycle decomposition* Prop. 10 p.125

$$(a_1\ a_2\ \ldots\ a_{k_1})(b_1\ b_2\ \ldots\ b_{k_2})\ldots$$

Then $\tau\sigma\tau^{-1}$ has cycle decomposition

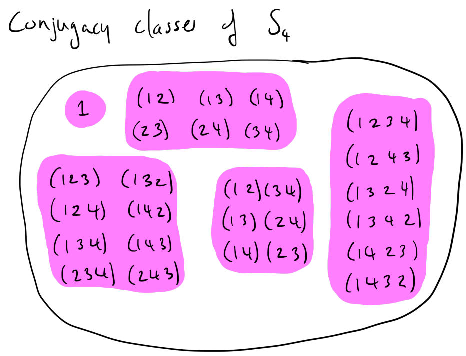
$$(\tau(a_1)\ \tau(a_2)\ \ldots\ \tau(a_{k_1}))(\tau(b_1)\ \tau(b_2)\ \ldots\ \tau(b_{k_2}))\ldots$$

Because of this, permutations in S_n are conjugate if and only if they have the same cycle type. Prop. 11 p.126

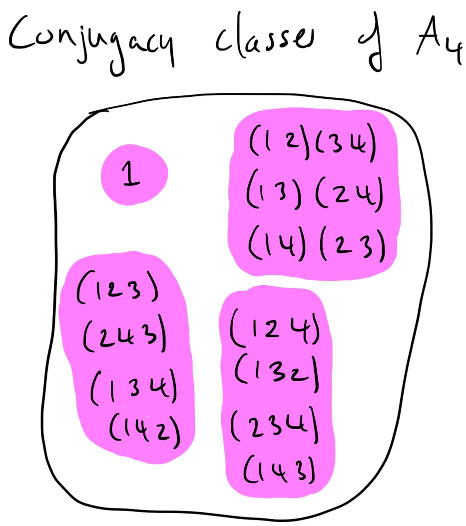
Definition. (1) If $\sigma \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$ with $n_1 \leq n_2 \leq \cdots \leq n_r$ (including 1-cycles) then the integers $n_1, n_2, \ldots, n_r$ are calledd the *cycle type* of σ.

(2) If $n \in \mathbb{Z}^+$, an *integer partition* of n is any nondecreasing sequence of positive integers whose sum is n.

For example, these are the conjugacy classes of S_4. Its possible cycle types are 1+1+1+1, 1+1+2, 2+2, 1+3, 4.



For the alternating group A_4, we have to be careful. Here permutations of the same cycle type need not be conjugate, since a permutation that conjugates one to the other need not be even. A_4 has the following cycle types.



Exercises

Exercise 1. List the elements of S_3 as 1, (1 2), (2 3), (1 3), (1 2 3), (1 3 2) and label these with integers 1,2,3,4,5,6 respectively. Exhibit the image of each element of S_3 under the **right** regular representation of S_3 into S_6.

Exercise 2. Suppose that the elements of S_4 are in some way labelled by the integers $1, 2, \ldots, 24$. Let S_4 act on itself by left multiplication and consider the associated homomoprhism into S_{24}. Find the cycle type of the image of (1 2)(3 4). (Note that the cycle type is invariant under the earlier choice of labelling.)

Exercise 3. Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to Q_8. 4.2.4

Exercise 4. Find all conjugacy classes and their sizes in D_8, $S_3 \times S_2$ and A_4.