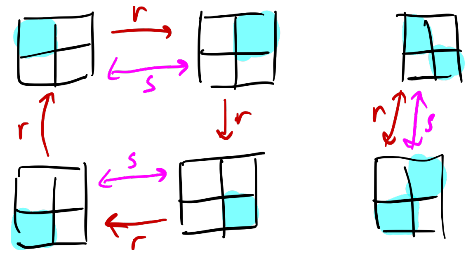
Group Actions and Orbits

We let a group G act on a set A. Sometimes, we may not be able to get from any set element to any other set element via the group action. For example, if we let D_8 act on the set below, we cannot turn a square with 2 blue quarters into a square with 1 blue quarter.



We call the connected sets *orbits*. More precisely, the orbit of an element $a \in A$ is defined as p.115

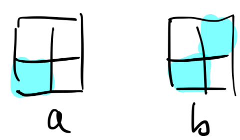
$$O_a = \{b \mid b = g \cdot a,\ g \in G\}.$$

These orbits turn out to partition the set A.

Proposition. *Let G be a group acting on the nonempty set A. The relation on A defined by $a \sim b$ if and only if $a = g \cdot b$ for some $g \in G$ is an equivalence relation. For each $a \in A$, the number of elements in the equivalence class containing a is $|G : G_a|$, the index of the stabilizer. In other words,* Prop. 2 p.115

$$|O_a| = |G : G_a|.$$

For example, consider the following two elements in the group action mentioned above.



Then $G_a = \{1, r^3 s\}$ and $|D_8 : G_a| = 4$, while $G_a = \{1, r^2, r^3 s, rs\}$ and $|D_8 : G_a| = 2$.

Exercises

Exercise 1. Let G be a group and $H \leq G$. Show that

$$N = \bigcap_{g \in G} gHg^{-1}$$

is a normal subgroup of G.

Exercise 2. Let G act on a set A. Prove that if $a, b \in A$ annd $b = g \cdot a$ for some $g \in G$, then $G_b = gG_a g^{-1}$. Deduce that if G acts transitively on A then the kernel of the action is $\bigcap_{g \in G} gG_a g^{-1}$. 4.1.1

Exercise 3. Show that the set of rigid motions of the tetrahedron is isomorphic to A_4. 3.5.7

Exercise 4. Show that, for all $n \geq 2$, 3.5.4

$$S_n = \langle (1\ 2),\ (1\ 2\ 3\ \ldots\ n) \rangle.$$