The Isomorphism Theorems

The Isomorphism Theorems are straightforwardd consequences of facts we have learned about subgroups and quotient groups. We will state them all here without proofs, and you will hopefully see how useful they are in future exercises.

Theorem (The First Isomorphism Theorem)**.** *If $\phi : G \to H$ be a homomorphism of groups, then $\ker \phi \trianglelefteq G$ and $G/\ker \phi \simeq \phi(G)$.* Thm. 16 p.97

Corollary. *Let $\phi : G \to H$ be a homomorphism of groups.* Cor. 17 p.97

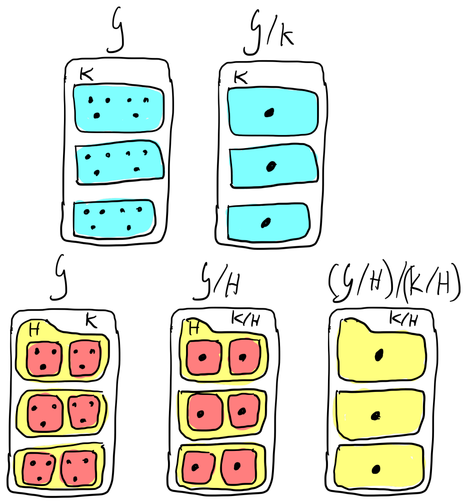
(i) ϕ *is injective if and only if* $\ker \phi = 1$.

(ii) $|G : \ker \phi| = |\phi(G)|$.

Theorem (The Second Isomorphism Theorem)**.** *Let G be a group, let A and B be subgroups of G and assume $A \leq N_G(B)$. Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \simeq A/A \cap B$.* Thm. 18 p.97

Theorem (The Third Isomorphism Theorem)**.** *Let G be a group and let H and K be normal subgroups of G with $H \leq K$. Then $K/H \trianglelefteq G/H$ and* Thm. 19 p.98

$$(G/H)/(K/H) \simeq G/K.$$



Theorem (The Fourth Isomorphism Theorem)**.** *Let G be a group and let N be a normal subgroup of G. Then there is a bijection from the set of subgroups A of G which contain N onto the set of subgroups $\overline{A} = A/N$ of G/N. This bijection has the following properties: for all $A, B \leq G$ with $N \leq A$ and $N \leq B$, we have* Thm. 20 p.99

(1) $A \leq B$ *if and only if* $\overline{A} \leq \overline{B}$,

(2) if $A \leq B$, *then* $|B : A| = |\overline{B} : \overline{A}|$,

(3) $\overline{\langle A, B\rangle} = \langle \overline{A}, \overline{B}\rangle$,

(4) $\overline{A \cap B} = \overline{A} \cap \overline{B}$,

(5) $A \trianglelefteq G$ *if and only if* $\overline{A} \trianglelefteq \overline{G}$.

Even/odd permutations and the alternating group

Every permutation can be written as a product of (not necessarily disjoint) 2-cycles, or *transpositions*. This product is not unique, unlike the usual disjoint cycle decomposition. Even the number of transpositions involved is not unique, since we can always multiply by the identity $(1\ 2)(2\ 1)$. It turns out, that the number of such transpositions $\mod 2$ is unique, however. p.107

Please read the proof of this in the book, which describes a surjective homomorphism from S_n ($n \geq 2$) to the multiplicative version of the cyclic group on 2 elements $\{\pm 1\}$. In class we discussed another view of this. Prop. 23 p.108

Definition. An *inversion* in a permutation $\sigma \in S_n$ is a pair $i, j \in \{1, 2, \ldots, n\}$ such that $i < j$ while $\sigma(i) > \sigma(j)$.

A single transposition gives rise to an odd number of inversions. Consider the transposition $(i\ j)$ with $i < j$. Any pair of elements that does not include i or j is not inverted by this permutation: if $k, l \in \{1, \ldots, n\} \setminus \{i, j\}$, then $\sigma(k) = k$ and $\sigma(l) = l$. Any pair of elements k, i or k, j with $k < i$ or $k > j$ is not inverted, since $\sigma(k) = k$ and $\sigma(i) > \sigma(j)$. For each k such that $i < k < j$, we obtain 2 inversions, namely i, k and i, j. Together with the inversion of i, j this gives an odd total number of inversions.

As an exercise, convince yourself that the composition of two permutations $\sigma\tau$ has an inversion number with the same parity as the sum of inversion numbers of σ and τ.

Rather than having to first write each permutation as a product of transpositions, we can read its parity directly from the disjoint cycle decomposition. Even cycles are odd and odd cycles are even. This is a bit confusing, but try to remember that transpositions are odd.

Proposition. *The permutation σ is odd if and only if the number of cycles of even length in a cycle decomposition is odd.* Prop. 25 p.110

We now have a partition of S_n into two classes: even and odd permutations. The even permutations form the kernel of the surjective homomorphism to $\{\pm 1\}$ mentioned previously, and therefore form a normaal subgroup. This shows us that, for example, we have the same number of even and odd permutations.

Definition. The *alternating group of degree n*, denoted A_n, is the set of even permutations on n elements.

Exercises

Exercise 1. Let A and B be groups. Show that $\{(a, 1) \mid a \in A\}$ is a normal subgroup of $A \times B$ and the quotient of $A \times B$ by this group is isomorphic to B. 3.1.37

Exercise 2. Show that if $n \in \mathbb{Z}^+$ and H is the unique subgroup of G of order n then $H \trianglelefteq G$. 3.2.5

Exercise 3. Let $H \leq G$ and define a relation $\sim$ on G by $a \sim b$ if and only if $ab^{-1} \in H$. Prove that this is an equivalence relation and describe the equivalence classes. 3.2.7

Exercise 4. Prove that if H is a normal subgroup of G of prime index p, then for all $K \leq G$ either 3.3.3

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

Exercise 5. Let M and N be normal subgroups of G such that $G = MN$. Prove that 3.3.7

$$G/(M \cap N) \simeq (G/M \times G/N).$$

Exercise 6. Prove that σ^2 is an even permutation for every permutation σ. 3.5.2