Quotient groups

Recall the following Theorem.

Theorem. *Let G be a group and K the kernel of some homomorphism of G to another group. Then the set of left cosets of K in G together with the operation defined by* Thm. 3 p.77

$$uK \circ vK = (uv)K$$

forms the group G/K. (This operation is independent of the choice of representatives u, v.)

For subgroups of G that form kernels of some homomorphism, the above suggests that the left (or right) cosets form a partition of G. This is true more generally for subgroups and their left/right cosets. Note that in general, the two partitions given by left and right cosets respectively need not be equal. They are equal if and only if the subgroup is the kernel of some homomorphism as we'll see in the final theorem of Section 3.1.

Proposition. *Let N be any subgroup of a group G. The set of left (or right) cosets of N in G form a partition of G.* Prop. 4 p.80

Finally, we see that subgroups for which we can take quotients have various equivalent characterizations that we are already familiar with. This often makes our life easy as we can use different techniques to prove that quotients or homomorphisms exist.

Definition. The element gng^{-1} is called the *conjugate* of $n \in N$ by G. The set gNg^{-1} is called the *conjugate* of N by G. The element g is said to *normalize* N if $g \in N_G(N)$. A subgroup N is called *normal* in G if $N_G(N) = G$. If so, we use the notation $N \trianglelefteq G$.

Theorem. *Let N be a subgroup of a group G. The following are equivalent:* Thm. 6 Prop. 7 p.82

(1) $N \trianglelefteq G$

(2) $N_G(N) = G$

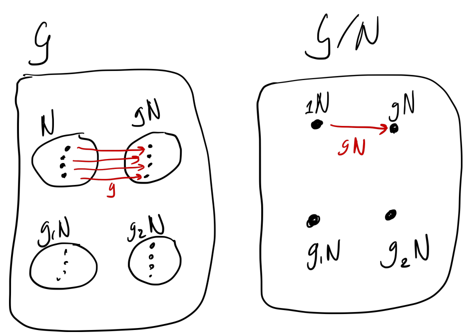
(3) $gN = Ng$ *for all* $g \in G$

(4) *the operation on left cosets of N defined by $uN \circ vN = (uv)N$ makes the set of left cosets into a group (same for right)*

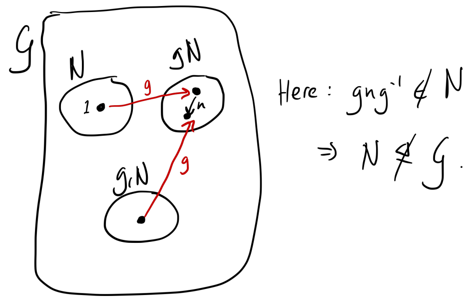
(5) $gNg^{-1} = N$ *for all* $g \in G$

(6) *N is the kernel of some homomorphism from G to another group.*

Visually, think of quotient groups as follows:



Again, visually, the above only works because the bundle of arrows corresponding to multiplying by g “agree” in that every element in gN is of the form ng with $n \in N$: they can be bundled together as the single arrow labeled gN from N to gN in the quotient group. When $gN \neq Ng$, this can be a problem, as shown below:



We may use Proposition 4 to say something even more precise about how the cosets of a subgroup partition the group: they partition the group *evenly*.

Theorem (Lagrange's Theorem)**.** *If G is a finite group and H is a subgroup of G, then the order of H divides the order of G, and the number of left (or right) cosets of H in G equals $|G|/|H|$.* Thm. 8 p.89

Proof. In addition to Proposition 4, all we need to prove this Theorem is the fact that all cosets have the same cardinality. We can do this by finding a bijection $H \to gH$ for any $g \in G$. The bijection $h \mapsto gh$ does the job. Write out the details yourself or check the proof in the book on page 90. □

Definition. If G is a group and $H \leq G$, the number of left cosets of H in G is called the *index* of H in G and is denoted $|G : H|$.

We obtain a couple of nice corollaries directly from Lagrange's Theorem.

Corollary. *If G is a finite group and $x \in G$, then the order of x divides the order of G. In particular $x^{|G|} = 1$ for all $x \in G$.* Cor. 9 p.90

Corollary. *If G is a group of prime order p, then G is cyclic.* Cor.10 p.90

The converse of Lagrange's Theorem is not true. If G is a group of order n and k divides n, there is not necessarily a subgroup of order k in G. We do have a few partial converses.

Theorem. *If G is an abelian and k divides $|G|$, then G has a subgroup of order k.*

Theorem (Cauchy's Theorem)**.** *If G is a finite group and p is a prime dividing $|G|$, then G has an element of order p.* Thm. 11 p.93

Theorem (Sylow)**.** *If G is a finite group of order $p^\alpha m$, where p is a prime and p does not divide m, then G has a subgroup of order p^α.* Thm. 12 p.93

Exercises

Exercise 1. Let $\phi : G \to H$ be a homomorphism with kernel K and let $a, b \in \phi(G)$. Let $X \in G/K$ be the fiber above a and Y the fiber above b. Fix an elemtn $u \in X$. Prove that if $XY = Z$ in the quotient group G/K and w any member of Z, then there is some $vinY$ such that $uv = w$. 3.1.2

Exercise 2. Let A be an abelian group and B a subgroup of A. Show that A/B is abelian. Give an example of a non-abelian group G with a proper normal subgroup N such that G/N is abelian. 3.1.3

Exercise 3. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that if $G = \langle x, y \rangle$ then $\overline{G} = \langle \overline{x}, \overline{y} \rangle$. (It is true more generally that if $G = \langle S \rangle$ for any subset S of G then $\overline{G} = \langle \overline{S} \rangle$.) 3.1.16

Exercise 4. Let $H \leq G$ and fix some element $g \in G$. Prove that gHg^{-1} is a subgroup of G of the same order as H. 3.2.5

Exercise 5. Let $H \leq G$ and let $g \in G$. Prove that if the right coset Hg equals *some* left coset of H in G then it equals the left coset gH and g must be in $N_G(H)$. 3.2.6

Exercise 6. Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = \{1\}$. 3.2.8