More on group actions

The following Lemma is crucial for understanding group actions. Each element of G induces a permutation on the set A. From looking Cayley diagrams and our understanding of groups, we know that for any element $g \in G$ the map $h \mapsto gh$ for $h \in G$ induces a permutation on the group itself. This is why in Cayley diagrams, each element has exactly one incoming and one outgoing arrow of each color. One difference with group actions is that in a group, we have that for each $a, b \in G$ there is a unique element g that "sends a to b" via $ag = b$, while with group actions in general, we may have that two elements $g \neq h \in G$ induce the same permutation on A.

Lemma. *For a given group G with action $\cdot$ on A, let $\sigma_g : A \to A$ be defined by $\sigma_g(a) = g \cdot a$ for all $a \in A$. Then* p.42

(1) for each fixed g, σ_g is a permutation of A,

(2) the map $\phi : G \to S_A$ given by $g \mapsto \sigma_g$ is a homomorphism.

Definition. We say that an action is *faithful* if $\sigma_g = \sigma_h$ implies that $g = h$. We define the *kernel* of the action as the set $\{g \in G \mid g \cdot a = a \text{ for all } a \in A\}$. Ex. (1) p.43

In the exercises, you show that actions are faithful if and only if the kernel is trivial. This is analoguous to linear algebra, where a linear transformation is invertible if and only if the kernel of the matrix is trivial.

Subgroups

Definition. Let G be a group and H a nonempty subset of G. H is a subgroup if it is closed under taking products and inverses. (The other properties are inherited from G.) In this case, we write $H \leq G$. p.46

Again, this is also something that might seem familiar to you from the topic of vector spaces and subspaces. We can summarize these two closure properties in one, although I personally most often find it easier to check them separately.

Proposition. *A subset H of a group G is a subgroup if and only if* Prop. 1 p.47

(1) $H \neq \emptyset$,

(2) for all $x, y \in H$, we have $xy^{-1} \in H$.

If H is finite, then it suffices to check that H is nonempty and closed under taking inverses.

Example. Consider the action of a group G on itself, via $g \cdot h = gh$. Then, it is easy to see that this action is faithful. Therefore, we have found an injective homomorphism $\phi : G \to S_G$, or to S_n if $|G| = n$ finite. The image of this homomorphism is a subgroup of S_n. (Show that this is true as an exercise.) And this subrgoup of S_n is isomorphic to G. Since we can do this in general, we see that the symmetric groups S_n contain the structure of all other groups as subgroups.

Exercises

Exercise 1. Show that the order of an element in S_n equals the least common multiple of lengths of the cycles in its cycle decomposition. (You may use the result from Exercise 1 in Homework 2.) 1.3.15

Exercise 2. Find all the numbers n such that S_6 contains and element of order n. 1.3.18

Exercise 3. Show that if $\phi : G \to H$ is a homomorphism, then we must have that $\phi(1) = 1'$, where 1 is the identity of G and $1'$ the identity of H.

Exercise 4. Show that the additive groups $\mathbb{Z}$ and $3\mathbb{Z}$ are isomorphic.

Exercise 5. Prove that a group G acts faithfully on a set A if and only if the kernel of the action is the set consisting only of the identity. 1.7.6

Exercise 6. Asssume n is an even positive integer and show that D_{2n} acts on the set consisting of pairs of opposite vertices of a regular n-gon. Find the kernel of this action. 1.7.12