Simple groups and composition series

Definition. A *simple group* is a group that has no non-trivial, proper normal subgroups. In other words, if G is simple and $H \trianglelefteq G$, then $H \in \{\langle 1 \rangle, G\}$.

We can think of simple groups somewhat analoguously to prime numbers: they are not "divisible" in any non-trivial way. Indeed, any group of prime order is simple, but this is not a necessary condition. We put together a lot of our tools so far to show the following result.

Theorem. *The alternating group A_n is simple for all $n \geq 5$.* Thm. 24 p149

The following notion is then analogous to prime decomposition of natural numbers.

Definition. In a group G a sequence of subgroups

$$1 = N_0 \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_{k-1} \trianglelefteq N_k = G$$

is called a *composition series* if $N_i \trianglelefteq N_{i+1}$ and N_{i+1}/N_i is a simple group for $0 \leq i \leq k-1$. If the above sequence is a composition series, then the quotient groups N_{i+1}/N_i are called composition factors of G.

Theorem (Jordan-Hölder)**.** *Let G be a finite group with $G \neq 1$. Then* Thm. 22 p103

(1) G has a composition series.

(2) The composition factors in a composition series are unique, namely, if $1 = N_0 \trianglelefteq N_1 \leq \cdots \trianglelefteq N_r = G$ and $1 = M_0 \trianglelefteq M_1 \trianglelefteq \cdots \trianglelefteq M_s = G$ are two composition series for G, then $r = s$ and there is some permutation $\pi \in S_r$ such that, for $0 \leq i \leq r-1$,

$$M_{\pi(i)+1}/M_{\pi(i)} \simeq N_{i+1}/N_i.$$

We will prove this theorem in class and in the exercises below.

Exercises

Exercise 1. Show that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$. 4.6.1

Exercise 2. Find all normal subgroups of S_n for all $n \geq 5$. 4.6.2

Exercise 3. Show that A_n is the only proper subgroup of index $< n$ in S_n for all $n \geq 5$. 4.6.3

Exercise 4. Prove part (1) of the Jordan-Hölder Theorem. 3.4.6

Exercise 5. If G is a finite group and $H \trianglelefteq G$, prove that there is a composition series of G, one of whose terms is H. 3.4.7

Exercise 6. Prove the following special case of part (2) of the Jordan-Hölder Theorem. Assume the finite group G has two composition series 3.4.9

$$1 = N_0 \trianglelefteq N_1 \leq \cdots \trianglelefteq N_r = G \quad \text{and} \quad 1 = M_0 \trianglelefteq M_1 \trianglelefteq M_2 = G.$$

Show that $r = 2$ and that the list of composition factors is the same.