Commutators and semi-direct products

Let G be a group and $x, y \in G$ be two elements. Then $[x, y] = x^{-1}y{-}1xy$ is called the *commutator* of x and y. Then $xy = yx[x, y]$, so in a sense, the commutator tells us what happens when x and y "commute". In particular, $[x, y] = 1$ if and only if x and y actually commute. We can use these commutators to prove the following helpful theorem for recognizing direct products.

Theorem. *Suppose G is a group with subgroups H and K, such that* Thm. 9 p.171

(1) H and K are normal in G,

(2) $H \cap K = 1$.

Then $HK \simeq H \times K$.

The full proof is in the book. Note an important part of it: we need that H and K are both normal in G in order to obtain that $h^{-1}k^{-1}hk \in H \cap K$, and therefore elements of H commute with elements of K.
It turns out that when only one of H and K is normal in G (and elements of h and k don't commute), we can still describe a version of products, called the semi-direct product.

Theorem. *Let H and K eb groups, and ϕ a homomorphism from K to $\mathrm{Aut}(H)$. Let $\cdot$ be the left action of K on H defined by ϕ. Let* Thm. 10 p.176

$$G = \{(h, k) \mid h \in H, k \in K\}$$

(just like we do with direct products) defined by the following multiplication operation:

$$(h_1, k_1)(h_2, k_2) = (h_1, k_1 \cdot h_2, k_1k_2).$$

This makes G a group of order $|G| = |H| \cdot |K|$, with (in the same way as we did with the direct products)

$$H \simeq \{(h, 1) \in G \mid h \in H\}, \quad K \simeq \{(1, k) \in G \mid k \in K\}.$$

Identifying H and K with their isomorphic copies, we have $H \trianglelefteq G$ and $H \cap K = 1$.

We write $G = H \rtimes_\phi K$. We can similarly recognize semi-direct products:

Theorem. *Let G be a group with subgroups H and K such that $H \trianglelefteq G$ and $H \cap K = 1$. Then $HK \simeq H \rtimes K$, where the left action of K on H is conjugation.* Thm. 12 p.180

In class, we looked at some further classifications by isomorphism types of groups of given orders. (Now including non-abelian groups as well as abelian groups.) We found, for example, that there are 5 groups of order 12 (2 abelian and 3 non-abelian). We also showed that

Proposition. *Let $p < q$ be two primes. If $p \not| q - 1$, then any group of order pq is abelian. If $p | q - 1$, there is at least one non-abelian group of order pq.*

See pages 181-183 in the book for these and further examples.

Exercises

For the last three exercises, let H, K be groups and let $\phi : K \to \mathrm{Aut}(H)$. Identify H and K as subgroups of $G : H \rtimes_\phi K$.

Exercise 1. Show that if $x, y \in G$ then $[y, x] = [x, y]^{-1}$. 5.4.1

Exercise 2. Let $a, b, c \in G$. Show that $[a, bc] = [a, c](c^{-1}[a, b]c)$ and that $[ab, c] = (b^{-1}[a, c]b)[b, c]$. 5.4.3

Exercise 3. Find the commutator subgroups of S_4 and A_4. 5.4.4

Exercise 4. Show that $C_G(H) \cap K = \ker \phi$. 5.5.1

Exercise 5. Show that $C_G(H) \cap K = N_G(H) \cap K$. 5.5.2

Exercise 6. Use semi-direct products to construct two groups of order 8, then show that both of these are isomorphic to D_8. 5.1.4