Commutators and semi-direct products

Let $G$ be a group and $x, y \in G$ be two elements. Then $[x, y] = x^{-1}y^{-1}xy$ is called the commutator of $x$ and $y$. Then $xy = yx[x, y]$, so in a sense, the commutator tells us what happens when $x$ and $y$ “commute”. In particular, $[x, y] = 1$ if and only if $x$ and $y$ actually commute.

We can use these commutators to prove the following helpful theorem for recognizing direct products.

**Theorem.** Suppose $G$ is a group with subgroups $H$ and $K$, such that

1. $H$ and $K$ are normal in $G$,
2. $H \cap K = 1$.

Then $HK \simeq H \times K$.

The full proof is in the book. Note an important part of it: we need that $H$ and $K$ are both normal in $G$ in order to obtain that $h^{-1}k^{-1}hk \in H \cap K$, and therefore elements of $H$ commute with elements of $K$.

It turns out that when only one of $H$ and $K$ is normal in $G$ (and elements of $h$ and $k$ don’t commute), we can still describe a version of products, called the semi-direct product.

**Theorem.** Let $H$ and $K$ be groups, and $\phi$ a homomorphism from $K$ to $\text{Aut}(H)$. Let $\cdot$ be the left action of $K$ on $H$ defined by $\phi$. Let

$$G = \{(h, k) \mid h \in H, k \in K\}$$

(just like we do with direct products) defined by the following multiplication operation:

$$(h_1, k_1)(h_2, k_2) = (h_1, k_1 \cdot h_2, k_1k_2).$$

This makes $G$ a group of order $|G| = |H| \cdot |K|$, with (in the same way as we did with the direct products)

$$H \simeq \{(h, 1) \in G \mid h \in H\}, \quad K \simeq \{(1, k) \in G \mid k \in K\}.$$

Identifying $H$ and $K$ with their isomorphic copies, we have $H \trianglelefteq G$ and $H \cap K = 1$.

We write $G = H \rtimes_\phi K$. We can similarly recognize semi-direct products:

**Theorem.** Let $G$ be a group with subgroups $H$ and $K$ such that $H \trianglelefteq G$ and $H \cap K = 1$. Then $HK \simeq H \rtimes K$, where the left action of $K$ on $H$ is conjugation.

In class, we looked at some further classifications by isomorphism types of groups of given orders. (Now including non-abelian groups as well as abelian groups.) We found, for example, that there are 5 groups of order 12 (2 abelian and 3 non-abelian). We also showed that

**Proposition.** Let $p < q$ be two primes. If $p \nmid q - 1$, then any group of order $pq$ is abelian. If $p|q - 1$, there is at least one non-abelian group of order $pq$.

See pages 181-183 in the book for these and further examples.
Exercises

For the last three exercises, let $H, K$ be groups and let $\phi : K \to \text{Aut}(H)$. Identify $H$ and $K$ as subgroups of $G : H \rtimes_{\phi} K$.

**Exercise 1.** Show that if $x, y \in G$ then $[y, x] = [x, y]^{-1}$.

**Exercise 2.** Let $a, b, c \in G$. Show that $[a, bc] = [a, c](c^{-1}[a, b]c)$ and that $[ab, c] = (b^{-1}[a, c]b)[b, c]$.

**Exercise 3.** Find the commutator subgroups of $S_4$ and $A_4$.

**Exercise 4.** Show that $C_G(H) \cap K = \ker \phi$.

**Exercise 5.** Show that $C_G(H) \cap K = N_G(H) \cap K$.

**Exercise 6.** Use semi-direct products to construct two groups of order 8, then show that both of these are isomorphic to $D_8$. 