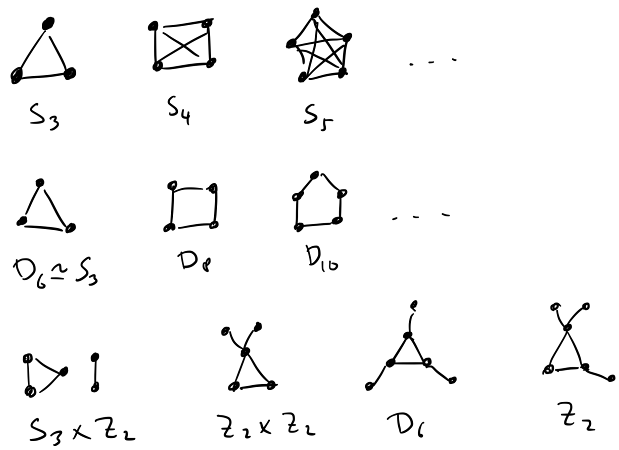
Automorphisms

As a warm-up to looking at automorphism groups of groups, we look at automorphism groups of graphs. A graph is a pair $H = (V, E)$ of a vertex set V and an edge set $E \subseteq \binom{V}{2}$ consisting of unordered pairs of vertices. An automorphism of a graph is a bijective map $\phi : V(H) \to V(H)$ such that $xy \in E(H)$ implies that $\phi(x)\phi(y) \in E(H)$ (and vice versa). For each of the following graphs, convince yourself that the automorphism groups are the ones listed.



One thing to note: the complete graph on 4 vertices can be drawn to look like a tetrahedron. However, when we considered symmetries of the tetrahedron, we considered it to be a fixed object in 3D space. The tetrahedron has symmetry group A_4, but the corresponding graph has automorphism group S_4 when not restricted in space.
Let G be a group. An isomorphism of G to itself is called an *autmorphism*. As an exercise, verify that the automorphisms of G form a group under composition. We call this group $\mathrm{Aut}(G)$.

As an example, consider the group $V_4 = \langle a, b \mid a^2 = b^2 = 1,\ ab = ba\rangle$. Note that in this group, any permutation of the three nontrivial elements gives a valid homomorphism. (They all have order 2 and combining any two distinct elements gives the third.) Therefore, $\mathrm{Aut}(G) = S_3$.

As another example of the automorphism group of abelian groups, we have the following.

Proposition. *The automorphism group of the cyclic group of order n is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.* Prop. 16 p135

We now focus our attention on a specific type automorphisms which does not occur in abelian groups: automorphisms obtained from conjugation.

Proposition. *Let H be a normal subgroup of a group G. Then G acts by conjugation on H as automorphisms of H. More specifically, the action of G on H by conjugation is defined for each $g \in G$ by* Prop. 13 p133

$$h \mapsto ghg^{-1}, \quad \textit{for each } h \in H.$$

for each $g \in G$, conjugation by g is an automorphism of H. The permutation representation afforded by this action is a homomorphism of G into $\mathrm{Aut}(H)$ *with kernel $C_G(H)$. In particular, $G/C_G(H)$ is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

Proposition. *For any subgroup H of a group G, the quotient group $N_G(H)/C_H(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$. In particular, $G/Z(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(G)$.* Prop. 15 p134

SageMath

SageMath is an open-source python-based language with a lot of specific functionality for mathematics. This course is not about programming, but not much is needed to be able to use SageMath to explore some of the important abstract algebra topics. I recommend running sage on your own machine, but you can also run short commands online (without needing to download anything) at `https://sagecell.sagemath.org`. Note that this is only one cell. If you ask for multiple pieces of output, it will only show the last one.

Here are some example commands. This defines the complete graph on 3 vertices as G, and shows a picture.

```
G=graphs.CompleteGraph(3)
G.show()
```

Now we can list the elements of its automorphim group, and ask whether its automorphism group is isomorphic to S_3. (It should be!)

```
S3=SymmetricGroup(3)
AG=G.automorphism_group()
AG.list()
AG.is_isomorphic(S3)
```

We add two vertices to G, both adjacent to 0. Now its automorphism group should be isomorphic to $Z_2 \times Z_2$.

```
G = graphs.CompleteGraph(3)
G.add_edge(0,3)
G.add_edge(0,4)
G.show()
Z2 = CyclicPermutationGroup(2)
Z2Z2 = cartesian_product([Z2,Z2])
AG = G.automorphism_group()
AG.is_isomorphic(Z2Z2)
```

Let's define the dihedral group D_{16}. This is called D_8 in SageMath. We can check the order of the group to be sure we have the right one.

```
D16 = DihedralGroup(8)
D16.order()
gap(D16).NormalSubgroups()
```

Finally, we define the symmetric group S_6, and a subgroup generated by two of its elements.

```
S6 = SymmetricGroup(6)
H = S6.subgroup(['(1,2)','(3,4)'])
```

Exercises

Exercise 1. If $\sigma \in \mathrm{Aut}(G)$ and ϕ_g is conjugation by g prove $\sigma\phi_g\sigma^{-1} = \phi_{\sigma(g)}$. Deduce that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$. 4.4.1

Exercise 2. Prove that if G is an abelian group of order pq, where p and q are distinct primes, then G is cyclic. 4.4.2

Exercise 3. Show that under any automorphism of D_8, r has at most 2 possible images and s has at most 4 possible images. Deduce that $|\,\mathrm{Aut}(D_8)| \leq 8$. 4.4.3

Exercise 4. Show that $|\,\mathrm{Aut}(Q_8)| \leq 24$. 4.4.4

Exercise 5. Use SageMath to show that $S_3 = \langle (1\ 2), (1\ 2\ 3) \rangle$. Show the input and output.

Exercise 6. Use SageMath and the lattice on p70 to identify all of the normal subgroups of D_{16}. Then compare this to D_{14}. Gather some more data to make a conjecture about the set of normal subgroups of D_{2n}.