Exercise 1. List the elements of S_3 as 1, (1 2), (2 3), (1 3), (1 2 3), (1 3 2) and label these with integers 1,2,3,4,5,6 respectively. Exhibit the image of each element of S_3 under the **right** regular representation of S_3 into S_6.

Solution. Of course, we have $\pi_1 = 1$. Next we see that

$$\begin{aligned}
\pi_2(1) &= 1(1\ 2) = (1\ 2) = 2 \\
\pi_2(2) &= (1\ 2)(1\ 2) = 1 \\
\pi_2(3) &= (2\ 3)(1\ 2) = (1\ 3\ 2) = 6 \\
\pi_2(4) &= (1\ 3)(1\ 2) = (1\ 2\ 3) = 5 \\
\pi_2(5) &= (1\ 2\ 3)(1\ 2) = (1\ 3) = 4 \\
\pi_2(6) &= (1\ 3\ 2)(1\ 2) = (2\ 3) = 3,
\end{aligned}$$

and therefore $\pi_2 = (1\ 2)(3\ 6)(4\ 5)$. Similarly, we see that $\pi_3 = (1\ 3)(2\ 5)(4\ 6)$, $\pi_4 = (1\ 4)(2\ 6)(3\ 5)$, $\pi_5 = (1\ 5\ 6)(2\ 3\ 4)$ and $\pi_6 = (1\ 6\ 5)(2\ 4\ 3)$.

Exercise 2. Suppose that the elements of S_4 are in some way labelled by the integers $1, 2, \ldots, 24$. Let S_4 act on itself by left multiplication and consider the associated homomoprhism into S_{24}. Find the cycle type of the image of (1 2)(3 4). (Note that the cycle type is invariant under the earlier choice of labelling.)

Solution. We see that

$$\begin{aligned}
1 &\mapsto (1\ 2)(3\ 4) \mapsto 1 \\
(1\ 2) &\mapsto (3\ 4) \mapsto (1\ 2) \\
(1\ 3) &\mapsto (1\ 4\ 3\ 2) \mapsto (1\ 3) \\
(1\ 4) &\mapsto (1\ 3\ 4\ 2) \mapsto (1\ 4) \\
(2\ 3) &\mapsto (1\ 2\ 4\ 3) \mapsto (2\ 3) \\
(2\ 4) &\mapsto (1\ 2\ 3\ 4) \mapsto (2\ 4) \\
(1\ 2\ 3) &\mapsto (2\ 4\ 3) \mapsto (1\ 2\ 3) \\
(1\ 3\ 2) &\mapsto (1\ 4\ 3) \mapsto (1\ 3\ 2) \\
(1\ 2\ 4) &\mapsto (2\ 3\ 4) \mapsto (1\ 2\ 4) \\
(1\ 3\ 4) &\mapsto (1\ 4\ 2) \mapsto (1\ 3\ 4) \\
(1\ 3\ 2\ 4) &\mapsto (1\ 4\ 2\ 3) \mapsto (1\ 3\ 2\ 4) \\
(1\ 4\ 2\ 3) &\mapsto (1\ 3\ 2\ 4) \mapsto (1\ 4\ 2\ 3)
\end{aligned}$$

The cycle type is $(2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2)$. (We could also guess this pattern immediately, given that (1 2)(3 4) is self-inverse.)

Exercise 3. Use the left regular representation of Q_8 to produce two elements of S_8 which generate a subgroup of S_8 isomorphic to Q_8. 4.2.4

Solution. We have $Q_8 = \langle i, j \rangle$. Let the elements of Q_8: $1, -1, i, j, k, -i, -j, -k$ be labeled as $1, 2, 3, 4, 5, 6, 7, 8$, respectively. Then, we have

$$\begin{aligned}
\pi_i(1) &= i \cdot 1 = 3 \\
\pi_i(2) &= i \cdot -1 = -i = 6 \\
\pi_i(3) &= i \cdot i = -1 = 2 \\
\pi_i(4) &= i \cdot j = k = 5 \\
\pi_i(5) &= i \cdot k = -j = 7 \\
\pi_i(6) &= i \cdot -i = 1 = 1 \\
\pi_i(7) &= i \cdot -j = -k = 8 \\
\pi_i(8) &= i \cdot -k = j = 4.
\end{aligned}$$

So, $\pi_i = (1\ 3\ 2\ 6)(4\ 5\ 7\ 8)$. Similarly $\pi_j = (1\ 4\ 2\ 7)(3\ 8\ 6\ 5)$. We have seen that these left representations are injective into the symmetric group. Therefore, $\langle (1\ 3\ 2\ 6)(4\ 5\ 7\ 8), (1\ 4\ 2\ 7)(3\ 8\ 6\ 5) \rangle \in S_8$ is isomorphic to D_8.