Exercise 1. Let G be a group and $H \leq G$. Show that

$$N = \bigcap_{g \in G} gHg^{-1}$$

is a normal subgroup of G.

Solution. This is one part of the proof of Theorem 4.2.3 on page 119.

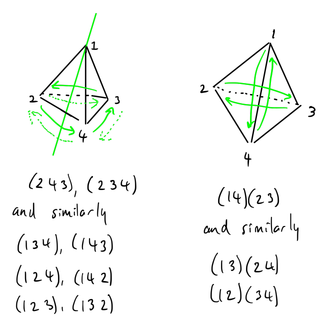
Exercise 2. Let G act on a set A. Prove that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then the kernel of the action is $\bigcap_{g \in G} gG_ag^{-1}$. 4.1.1

Solution. We have

$$\begin{aligned} G_b &= \{h \in G \mid h \cdot b = b\} \\ &= \{h \in G \mid h \cdot (g \cdot a) = g \cdot a\} \\ &= \{h \in G \mid (hg) \cdot a = g \cdot a\} \\ &= \{h \in G \mid g^{-1}(hg) \cdot a = g^{-1}g \cdot a\} \\ &= \{h \in G \mid (g^{-1}hg) \cdot a = a\} \\ &= \{h \in G \mid g^{-1}hg \in G_a\} \\ &= \{h \in G \mid h \in gG_ag^{-1}\} \\ &= gG_ag^{-1}. \end{aligned}$$

Exercise 3. Show that the set of rigid motions of the tetrahedron is isomorphic to A_4. 3.5.7

Solution. The tetrahedron has two types of symmetries (in addition to the identity): a rotation of 120° or 240° around an axis that passes through one of the corners and the center of the opposite face (there are 4 such axes), and a rotarion of 180° around an axis that passes through the centers of two opposite edges (3 such axes). This gives us indeed all even permutations of S_4:



Exercise 4. Show that, for all $n \geq 2$, 3.5.4

$$S_n = \langle (1\ 2),\ (1\ 2\ 3\ \ldots\ n) \rangle.$$

Solution. We have seen in Section 3.5 that all permutations can be written as products of transpositions. So, all we need to show is that we can obtain all transpositions in this manner. First we show that any transposition of the form $(i\ i+1)$, $1 \leq i \leq n-1$ can be generated by $(1\ 2)$, $(1\ 2\ 3\ \ldots\ n)$. We have

$$(1\ 2\ 3\ \ldots\ n)^{i-1}(1\ 2)(1\ 2\ 3\ \ldots\ n)^{-(i-1)} = (i\ i+1).$$

Now, we can achieve any transposition $(i\ j)$, by a series of adjacent transpositions: fist i moves to the position of j by swapping with $i+1, i+2, \ldots, j$ and then j moves to the position of i by swapping with $j-1, \ldots, i+1$. Therefore, we have

$$(i\ j) = (i\ i{+}1)(i{+}1\ i{+}2)\ldots(j{-}2\ j{-}1)(j{-}1\ j)(j{-}2\ j{-}1)\ldots(i{+}1\ i{+}2)(i\ i{+}1).$$