Exercise 1. Let A and B be groups. Show that $\{(a, 1) \mid a \in A\}$ is a normal subgroup of $A \times B$ and the quotient of $A \times B$ by this group is isomorphic to B. 3.1.37

Solution. Let's answer both questions at once. Let $A' = \{(a, 1) \mid a \in A\}$. Let $\phi : A \times B \to B$ be the homomorphism defined by $(a, b) \mapsto b$. (If needed, check explicitly that this is a group homomorphism). Then it is easy to see that $\ker \phi = A'$. By the definition of $A \times B$, for every $b \in B$ there is an element in $A \times B$ that is mapped to b by ϕ. For example, the element $(1, b) \in A \times B$. Therefore, ϕ is surjective and the result follows from the First Isomoprhism Theorem.

Exercise 2. Show that if $n \in \mathbb{Z}^+$ and H is the unique subgroup of G of order n then $H \trianglelefteq G$. 3.2.5

Solution. For every $g \in G$, we have gHg^{-1} is a subgroup of G with $|H| = |gHg^{-1}|$. Therefore if H is the unique subgroup of G of order n we must have that $H = gHg^{-1}$ for every $g \in G$, which is equivalent to $H \trianglelefteq G$ (by definition or by Theorem 6 p.82).

Exercise 3. Let $H \leq G$ and define a relation $\sim$ on G by $a \sim b$ if and only if $ab^{-1} \in H$. Prove that this is an equivalence relation and describe the equivalence classes. 3.2.7

Solution. We need to check the three properties of an equivalence relation:

(reflexive) For any $a \in G$, we have $aa^{-1} = 1 \in H$, since H is a subgroup of G.

(symmetric) Suppose that $a, b \in G$ and that $a \sim b$. Then $ab^{-1} \in H$. Since H is a subgroup, we have $(ab^{-1})^{-1} = ba^{-1} \in H$, which implies that $b \sim a$.

(transitive) Suppose that $a, b, c \in G$ and that $a \sim b, b \sim c$. Then $ab^{-1}, bc^{-1} \in H$. Since H is a subgroup, we have $ab^{-1}, bc^{-1} = ac^{-1} \in H$, which implies that $a \sim c$.

We have seen that $a \sim b$ iff $ab^{-1} \in H$ iff $Hab^{-1} = H$ iff $Ha = Hb$. Therefore $a \sim b$ when a and b are elements of the same right coset of H. In other words, the equivalence classes are the right cosets of H.

Exercise 4. Prove that if H is a normal subgroup of G of prime index p, then for all $K \leq G$ either 3.3.3

(i) $K \leq H$ or

(ii) $G = HK$ and $|K : K \cap H| = p$.

Solution. Suppose that $K \not\leq H$. Then there is some $k \in K \setminus H$. Then kH is a nonidentity element in G/H. Since G/H has prime order, it is generated by any non-identity element (since $\langle gH \rangle$ has order 1 or $p = |G/H|$ for any $g \in G$, and the former case is only when $gH = H$ is the identity element). Therefore $G/H = \langle kH \rangle$, and any element in G/H can be written as $k^i h$ for $h \in H$. This shows that $G = KH = HK$. Then, by the Second Isomorphism Theorem we have that $G/H \simeq K/K \cap H$, which shows that $|K : K \cap H| = |G : H| = p$.

Exercise 5. Let M and N be normal subgroups of G such that $G = MN$. Prove that 3.3.7

$$G/(M \cap N) \simeq (G/M \times G/N).$$

Solution. We will show this by finding a surjective homomorphism $\phi : G \to G/M \times G/N$ with kernel $M \cap N$. Then the result follows from the First Isomorphism Theorem. Let ϕ be defined by $g \mapsto (gM, gN)$. (It is easy to check that this is indeed a homomorphism.) Then $g \in \ker \phi$ if $g \mapsto (1M, 1N)$, i.e. if and only if $g \in M \cap N$. Therefore $\ker \phi = M \cap N$. To show that this map is surjective, consider any $(g_1M, g_2N) \in G/M \times G/N$. Then $g_1 = m_1n_1$ and $g_2 = m_2n_2$ for some $m_1, m_2 \in M$ and $n_1, n_2 \in N$, since $G = MN$. Let $g = n_1m_2$. Then $gM = n_1m_2M = n_1M = g_1M$ and $gN = Ng = Nn_1m_2 = Nm_2 = m_2N = g_2N$. Therefore, this homomorphism is surjective as needed.

Exercise 6. Prove that σ^2 is an even permutation for every permutation σ. 3.5.2

Solution. There are different ways of answering this question. If we use the view of permutation parity that we described in class, we saw that every permutation can be written as a product of transpositions, and that the parity of the number of transpositions in such a representation is fixed and determines the parity of the permutation. The result follows.