Exercise 1. Let $\phi : G \to H$ be a homomorphism with kernel K and let $a, b \in \phi(G)$. Let $X \in G/K$ be the fiber above a and Y the fiber above b. Fix an element $u \in X$. Prove that if $XY = Z$ in the quotient group G/K and w any member of Z, then there is some $v \in Y$ such that $uv = w$. 3.1.2

Solution. We have that X, Y, Z are all cosets of K. Fix an element $u \in X$ and $w \in Z$. Then $X = uK$ and $Z = wK$. Now $XY = Z$ and therefore $Y = X^{-1}Z$. Therefore, $Y = u^{-1}KwK = (u^{-1}w)K$. Then, $u^{-1}w \in Y$ and clearly $uu^{-1}w = w$.

Exercise 2. Let A be an abelian group and B a subgroup of A. Show that A/B is abelian. Give an example of a non-abelian group G with a proper normal subgroup N such that G/N is abelian. 3.1.3

Solution. Since A is abelian, we have for any $x, y \in A$ that $(xB)(yB) = xyB = yxB = (yB)(xB)$, which implies that A/B is abelian.

We have seen that $D_6/\langle r\rangle \simeq Z_2$, and therefore $D_6/\langle r\rangle$ is abelian even though D_6 is not.

Exercise 3. Let G be a group, let N be a normal subgroup of G and let $\overline{G} = G/N$. Prove that if $G = \langle x, y\rangle$ then $\overline{G} = \langle \overline{x}, \overline{y}\rangle$. (It is true more generally that if $G = \langle S\rangle$ for any subset S of G then $\overline{G} = \langle \overline{S}\rangle$.) 3.1.16

Solution. Suppose that $G = \langle x, y\rangle$, and consider the subgroup $\langle \overline{x}, \overline{y}\rangle$ of $\overline{G}$. By the Fourth Isomorphism Theorem, this subgroup is of the form $\overline{A} = A/N$ for a subgroup A of G containing N. Since, by definition, $\overline{x}, \overline{y} \in \overline{A}$, we therefore have $x, y \in A$. The subgroup $\langle x, y\rangle$ is defined as the smallest (by set inclusion) subgroup of G that contains x, y, so therefore we must have that $A = G$, which implies that $\langle \overline{x}, \overline{y}\rangle = \overline{A} = \overline{G}$.

Exercise 4. Let $H \le G$ and fix some element $g \in G$. Prove that gHg^{-1} is a subgroup of G of the same order as H. 3.2.5

Solution. Suppose that $x, y \in gHg^{-1}$. Then we have $x = gh_1g^{-1}$ and $y = gh_2g^{-1}$ for some $h_1, h_2 \in H$. We have

$$\begin{aligned} xy^{-1} &= gh_1g^{-1}(gh_2g^{-1})^{-1} \\ &= gh_1g^{-1}gh_2^{-1}g^{-1} \\ &= gh_1h_2^{-1}g^{-1} \\ &\in gHg^{-1}, \end{aligned}$$

since $h_1h_2^{-1} \in H$ (H is a subgroup). By the subgroup criterion, we conclude that gHg^{-1} is a subgroup of G.

To show that this subgroup has the same order as H, we let $f : H \to gHg^{-1}$ be given by $h \mapsto ghg^{-1}$. By left/right cancellation we have that $h_1 = h_2 \iff gh_1g^{-1} = gh_2g^{-1}$. Therefore, f is one-to-one and in fact invertible by letting f^{-1} be given by $ghg^{-1} \mapsto g^{-1}(ghg^{-1})g = h$. Therefore, $|H| = |gHg^{-1}|$.

Exercise 5. Let $H \le G$ and let $g \in G$. Prove that if the right coset Hg equals *some* left coset of H in G then it equals the left coset gH and g must be in $N_G(H)$. 3.2.6

Solution. Suppose that $Hg = g_1H$ for some $g_1 \in G$. Since $1 \in H$, we have $1g = g \in Hg$, and since $Hg = g_1H$ we have $g \in g_1H$ and therefore $g_1H = gH$. Then, we have $Hg = gH$, which implies that $H = gHg^{-1}$ and $g \in N_G(H)$.

Exercise 6. Prove that if H and K are finite subgroups of G whose orders are relatively prime, then $H \cap K = \{1\}$. 3.2.8

Solution. By Lagrange, we have that $H \cap K$ is a subgroup of both H and K and therefore its order divides both the order of H and K. Since the orders of H and $K$are relatively prime we must have $|H \cap K| = 1$ and therefore $H \cap K = \{1\}$.