Exercise 1. Find all cyclic subgroups of D_8. Find a proper subgroup of D_8 which is not cyclic. 2.3.11

Solution. All reflections have order 2, and therefore each generate distinct subgroups. The basic rotation r has order 4, and therefore $\langle r\rangle = \langle r^3\rangle$. Finally, r^2 generates another cyclic subgroup of order 2. Therefore, we have the cyclic subgroups

$$\langle r\rangle,\ \langle r^2\rangle,\ \langle s\rangle, \langle sr\rangle,\ \langle sr^2\rangle,\ \langle sr^3\rangle.$$

An example of a non-cyclic subgroup is $\langle s, r^2\rangle$. This group has elements $\{1, s, r^2, sr^2\}$ and is therefore not equal to D_8. Since it has 4 elements that have order 1 or 2, it is not cyclic.

Exercise 2. Show that if H is any group and h is an element of H, then there is a unique homomorphism from $\mathbb{Z}$ to H such that $1 \mapsto h$. 2.3.19

Solution. Suppose that $\phi : \mathbb{Z} \to H$ is a homomorphism of groups, such that $\phi(1) = h$. Then for any $n \in \mathbb{Z}^+$, we have

$$\phi(n) = \phi(1^n) = \phi(1)^n = h^n.$$

In general, a homomorphism of a cyclic group is fully determined by the image of a generator.

Exercise 3. Prove that if A is a subset of B, then $\langle A\rangle \leq \langle B\rangle$. Give an example where $A \subset B$ (proper subset) but $\langle A\rangle = \langle B\rangle$. 2.4.2

Solution. We have that

$$\langle A\rangle = \bigcap_{A\subseteq H,\ H\leq G} H \leq \bigcap_{B\subseteq H,\ H\leq G} H = \langle B\rangle,$$

since $A \subseteq B$. In other words, $\langle A\rangle$ is a subset of every H that contains A, and since every subgroup that contains B also contains A, $\langle A\rangle$ is a subset of every H that contains B.
For example, consider the group $\mathbb{Z}$. Then $\langle 2\rangle = \langle 2, 4\rangle$.

Exercise 4. Prove that the subgroup of S_4 generated by (1 2) and (1 2)(3 4) is a noncyclic group of order 4. 2.4.6

Solution. We can write out the multiplication table. Note that this subgroup is abelian.

	1	(1 2)	(3 4)	(1 2)(3 4)
1	1	(1 2)	(3 4)	(1 2)(3 4)
(1 2)	(1 2)	1	(1 2)(3 4)	(3 4)
(3 4)	(3 4)	(1 2)(3 4)	1	(1 2)
(1 2)(3 4)	(1 2)(3 4)	(3 4)	(1 2)	1

Note that all non-identity elements of the group have order 2 and therefore the group is not cyclic. You might also note that this subgroup is isomorphic to the abelian group of order for that we found in Exercise 1: $\{1, s, r^2, sr^2\}$ in D_8.

Exercise 5. Find all elements $x \in D_{16}$ such that $D_{16} = \langle x, s\rangle$. (There are 8 such elements.) 2.5.5

Solution. It is helpful to use the subgroup lattice on page 70. We know that $D_{16} = \langle r, s\rangle$. So it is sufficient if x and s can generate any r^k with k coprime to 8. This gives the 8 possible values for $x : r,\ r^3,\ r^5,\ r^7, sr,\ sr^3,\ sr^5,\ sr^7$.

Exercise 6. Let $\phi : G \to H$ be a homomorphism and let E be a subgroup of H. Prove that $\phi^{-1}(E) \leq G$. 3.1.1

Solution. Let $N = \phi^{-1}(E)$. Since $1_H \in E$, we must have $1_G \in N$. Suppose that $x, y \in N$, with $\phi(x) = a, \phi(y) = b \in E$. Then $\phi(xy^{-1}) = \phi(x)\phi(y)^{-1} = ab^{-1}$. Since $E \leq H$, we have $ab^{-1} \in E$, and therefore $xy^{-1} \in N$.