Exercise 1. Show that for any subset A of a group G, $N_G(A)$ is a subgroup of G.

Solution. We use Proposition 2.1.1 as usual. Since $1a1^{-1} = a$ for any $a \in A$, We have that $\{1a1^{-1} \mid a \in A\} = A$ and therefore $1 \in N_G(A)$ which shows that $N_G(A) \neq \emptyset$. Suppose that $x, y \in N_G(A)$. Note that $yAy^{-1} = A$ implies that $y^{-1}Ay = A$. Then, we have
Then we have, for any $a \in A$, that

$$(xy^{-1})A(xy^{-1})^{-1} = x(y^{-1}Ay)x^{-1} = xAx^{-1} = A,$$

And therefore $xy^{-1} \in N_G(A)$, which completes the proof.

Exercise 2. Show that $C_G(Z(G)) = G$ and deduce that $N_G(Z(G)) = G$. 2.2.2

Solution. Let $z \in Z(G)$ and $g \in G$. Then $zgz^{-1} = g$. Multiplying on the right by zg^{-1} gives $z = gzg^{-1}$, which implies that $g \in C_G(Z(G))$ for any $g \in G$. Therefore, $C_G(Z(G)) = G$. We have seen that $C_G(A) \leq N_G(A) \leq G$ for any $A \subseteq G$. In this case, this shows that $C_G(Z(G)) = N_G(Z(G)) = G$.

Exercise 3. Let H be a subgroup of G. 2.2.6

(a) Show that $H \leq N_G(H)$. Give an example to show that this is not necessarily true if H is not a subgroup.

(b) Show that $H \leq C_G(H)$ if and only if H is abelian.

Solution.

(a) Suppose that $H \leq G$, and let $h \in H$. Since H is closed under the group operation, we have that $hHh^{-1} \subseteq H$. Similarly, for any $k \in H$, we have that $h^{-1}kh \in H$, and therefore $k = h(h^{-1}kh)h^{-1} \in hHh^{-1}$. This shows that $H \subseteq hHh^{-1}$, and therefore $hHh^{-1} = H$. We deduce that $H \leq N_G(H)$.

The notation $H \leq N_G(H)$ implies that H is a subgroup, so this holds trivially. However, if H is not a subgroup, then $H \subseteq N_G(H)$ is not even true necessarily. For example, let $H = \{r, s\}$ in D_6. Then $rHr^{-1} = \{r, r^2s\} \neq H$, and therefore $H \not\subseteq N_G(H)$.

(b) Suppose that H is abelian. Then for any $h, k \in H$, we have that $hk = kh$. Multiplying on the right by h^{-1}, yields $hkh^{-1} = k$ which implies that $H \leq C_G(H)$.

If H is not abelian, then there is some pair $h, k \in H$ such that $hk \neq kh$. Multiplying on the right by h^{-1}, yields $hkh^{-1} \neq k$. Therefore there must be some $h \in H$ such that $h \notin C_G(H)$ and therefore $H \not\leq C_G(H)$.

Exercise 5. Find all generators for $\mathbb{Z}/202\mathbb{Z}$. 2.3.4

Solution. By Proposition 2.3.6, we have that $x \in \mathbb{Z}/202\mathbb{Z}$ is a generator if and only if $(x, 202) = 1$. The prime factorization of 202 is 2×101, so x can be any odd number between 1 and 201 except for 101. Or, by computer, we find

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sage: [k for k in range(202) if gcd(k,202)==1]
[1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37,
 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73,
 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99, 103, 105, 107, 109,
 111, 113, 115, 117, 119, 121, 123, 125, 127, 129, 131, 133, 135, 137,
 139, 141, 143, 145, 147, 149, 151, 153, 155, 157, 159, 161, 163, 165,
 167, 169, 171, 173, 175, 177, 179, 181, 183, 185, 187, 189, 191, 193,
 195, 197, 199, 201]
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Exercise 6. If x is an element of a finite group G and $|x| = |G|$, show that $G = \langle x \rangle$. Give an example to show that this need not be true if G is an infinite group. 2.3.2

Solution. By Proposition 2.3.2, we have that $|\langle x \rangle| = |x|$. Therefore, if G is finite, $|\langle x \rangle| = |G|$ and $\langle x \rangle \subseteq G$implies that $\langle x \rangle = G$.
Let G be the $\mathbb{Z}$ under addition. Then $|G| = \infty$. Then $\langle 2 \rangle = 2\mathbb{Z}$ and $|2| = |G| = \infty$. However $2\mathbb{Z} \neq \mathbb{Z}$.