Exercise 1. Prove that if σ is the m-cycle $(a_1\ a_2\ \ldots\ a_m)$, then for all $1 \leq k \leq m$, $\sigma^i(a_k) = a_{k+i}$, where $k+i$ is replaced by its least positive residue $\pmod m$. Deduce that $|\sigma| = m$. 1.3.10

Solution. We have that

$$\begin{aligned}
\sigma^i(a_k) &= \sigma^{i-1}(\sigma(a_k)) \\
&= \sigma^{i-1}(a_{k+1}) \\
&= \sigma^{i-2}(\sigma(a_{k+1})) \\
&= \sigma^{i-2}(a_{k+2}) \\
&\vdots \\
&= a_{k+i}
\end{aligned}$$

where each index is replaced by its least positive residue $\pmod m$. This implies that for $1 \leq k \leq m$, we have that m is the smallest positive integer value for i such that $\sigma^i(a_k) = a_k$. Therefore, $|\sigma| = m$.

Exercise 2. Let $\phi : G \to H$ be a homomorphism. Prove that $\phi(x^n) = \phi(x)^n$ for all $n \in \mathbb{Z}$. 1.6.1

Solution. We can show this by induction. Since ϕ is a homomorphism, we have that $\phi(ab) = \phi(a)\phi(b)$, for all $a, b \in G$. The case for $n = 0$ and $n = 1$ are straightforward. Suppose that the statement is true for $1, \ldots, n-1$. Then

$$\phi(x^n) = \phi(x^{n-1}x) = \phi(x^{n-1})\phi(x) = \phi(x)^{n-1}\phi(x) = \phi(x)^n,$$

as desired. For negative values of n, we note that the statement holds for any element of G, and therefore we may replace x by x^{-1}. Since $\phi(1) = 1$, we have that $\phi(xx^{-1}) = \phi(x)\phi(x^{-1}) = 1$, and therefore $\phi(x^{-1}) = \phi(x)^{-1}$. The result follows.
Finally, for the case $n = 0$, we now have, for any $x \in G$

$$\phi(1) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}) = \phi(x)\phi(x)^{-1} = 1.$$

Exercise 3. Show that if $\phi : G \to H$ is a homomorphism, then $|x| = k$ implies that $|\phi(x)|$ divides k. 1.6.2

Solution. We first need the following claim. For any group H and element $a \in H$, we have that $a^k = 1$ implies that $|a|$ divides k. Suppose that $|a| = l$ and does not divide k. Then $1 < l < k$. Let q be the greatest integer such that $ql < k$. Since l does not divide k, we have that $k - ql < l$. However, $x^{k-ql} = 1$, since $x^k = x^{ql} = 1$. This contradicts the fact that l is the order of x.
Homomorphisms must map identity elements to identity elements. If you don't show that carefully this time, it will be in the next homework. Then, $\phi(x^k) = \phi(1) = 1$, and also $\phi(x^k) = (\phi(x))^k = 1$. The result follows.

Exercise 4. Show that D_{24} and S_4 are not isomorphic. 1.6.9

Solution. The group D_{24} has an element of order 6, namely r^2. The group S_4 has elements of order 2 (those of the form $(a\ b)$ and $(a\ b)(c\ d)$), order 3 (those of the form $(a\ b\ c)$) and order 4 (those of the form $(a\,b\,c\,d)$), and none of order 6.
By the previous question and the fact that an isomorphism is a bijective homomorphism, we see that if ϕ is an isomorphism then $|\phi(x)| = |x|$ for all x. Therefore, if $\phi : D_{24} \to S_4$ were an isomorphism, we would have $|\phi(r^2)| = 6$, which is not possible.