Exercise 1. Show that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$. 4.6.1

Solution. We have seen that for all $n \geq 5$, A_n does not have a proper normal subgroup. Suppose that H is a proper subgroup of A_n of index $|A_n : H| = k < n$. Consider the action of A_n on H given by left multiplication. This gives rise to a homomorphism $\phi : A_n \to S_k$. However, since $n > 2$, we have that $|A_n| > |S_k|$, implying that $\ker \phi \neq 1$. Since A_n is simple, this implies that $\ker \phi = A_n$ which in turn implies that $H = A_n$.

Exercise 2. Find all normal subgroups of S_n for all $n \geq 5$. 4.6.2

Solution. Since A_n is a normal subgroup of S_n and A_n is simple, there are no normal subgroups of S_n of order less than $|A_n|$, or index greater than 2. Are there further normal subgroups of index 2? No. Suppose that there is another normal subgroup H. Then $H \cap A_n$ must be normal in A_n. This implies that either $H \cap A_n = 1$ or $H \cap A_n = A_n$. If $H \cap A_n = 1$ then H contains all but one of the odd permutations. For example, H contains a 4-cycle $\sigma = (a, b, c, d)$. But then $\sigma^2 \in H$ and $\sigma^2 \in A_n \setminus \{1\}$, a contradiction. Therefore $H \cap A_n = A_n$ and $H = A_n$.

Exercise 3. Show that A_n is the only proper subgroup of index $< n$ in S_n for all $n \geq 5$. 4.6.3

Solution. As before, suppose that H is a proper subgroup of S_n with index $2 < |S_n : H| = k < n$. Let the S_n act on the cosets of H by left multiplication, and consider the corresponding homomorphism $\phi : S_n \to S_k$. Since $k < n$ we cannot have $\ker \phi = 1$, and since $H \neq S_n$ we do not have $\ker \phi = S_n$. Since we must have $\ker \phi \leq H$, we do not have $\ker \phi = A_n$. Since this exhausts all normal subgroups of S_n, H does not exist.