Exercise 1. Show that if $x, y \in G$ then $[y, x] = [x, y]^{-1}$. 5.4.1

Solution. We have

$$[y, x] = y^{-1}x{-}1yx = (x^{-1}y{-}1xy)^{-1} = [x, y]^{-1}.$$

Exercise 2. Let $a, b, c \in G$. Show that $[a, bc] = [a, c](c^{-1}[a, b]c)$ and that $[ab, c] = (b^{-1}[a, c]b)[b, c]$. 5.4.3

Solution. We have

$$\begin{aligned}[a, bc] &= a^{-1}(bc)^{-1}abc \\ &= a^{-1}c^{-1}b^{-1}abc \\ &= a^{-1}c^{-1}b^{-1}ba[a, b]c \\ &= a^{-1}c^{-1}acc^{-1}[a, b]c \\ &= [a, c](c^{-1}[a, b]c),\end{aligned}$$

and

$$\begin{aligned}[ab, c] &= (ab)^{-1}c^{-1}abc \\ &= b^{-1}a^{-1}c^{-1}abc \\ &= b^{-1}a^{-1}c^{-1}acb[b, c] \\ &= [ab, c] = (b^{-1}[a, c]b)[b, c].\end{aligned}$$

Exercise 3. Find the commutator subgroups of S_4 and A_4. 5.4.4

Solution. First, note that all commutators are even permutations in S_4. Therefore, their cycle types are 3 or $2+2$. Consider any 3-cycle (x, y, z). Then

$$[(x, y), (x, z)] = (x, y)^{-1}(x, z)^{-1}(x, y), (x, z) = (x, y, z).$$

Therefore, all 3-cycles are in the commutator subgroup. Now consider any permutation of the form $(x, y)(z, w)$. Then

$$[(x, w, z), (w, z, y)] = (x, w, z)^{-1}(w, z, y)^{-1}(y, w, z)(w, z, y) = (x, y)(z, w),$$

and therefore all of those permutations are also in the commutator subgroup. We see that the commutator subgroup of S_4 is all of A_4.
A_4 has elements of the cycle type 3 or $2+2$. We have that

$$[(x, y, z), (x, y)(z, w)] = (x, y, z)^{-1}(z, w)^{-1}(x, y)^{-1}(x, y, z)(x, y)(z, w) = (x, w)(y, z),$$

and

$$[(x, z)(y, w), (x, y)(z, w)] = (y, w)^{-1}(x, z)^{-1}(z, w)^{-1}(x, y)^{-1}(x, z)(y, w)(x, y)(z, w) = 1.$$

By symmetry, we have now checked every possibility. Therefore, the commutator subgroup of A_4 is the set of elements consisting of the identity and all elements of the form $(x, y)(w, z)$.

Exercise 4. Show that $C_G(H) \cap K = \ker \phi$. 5.5.1

Solution. We have $G = H \rtimes_\phi K$.Suppose that $(1, x) \in C_G(H) \cap K$ and $(h, 1) \in H$. Then $(h, 1)(1, x) = (h, x)$ and $(1, x)(h, 1) = (x \cdot h, x)$. Since $(h, 1)(1, x) = (1, x)(h, 1)$, we have that $x \cdot h = h$ for all $h \in H$ ad therefore $x \in \ker \phi$.
Conversely, suppose that $(1, x) \in \ker \phi$. Then, by a similar (reverse) argument, we must have that $(1, x) \in C_G(H) \cap K$.

Exercise 5. Show that $C_G(K) \cap H = N_G(K) \cap H$. 5.5.2

Solution. Clearly, we have $C_G(K) \cap H \subseteq N_G(K) \cap H$. Let $(y, 1) \in N_G(K) \cap H$. Then for any $(1, k) \in K$, we have

$$(y, 1)(1, k)(y, 1)^{-1} = (y(k \cdot y^{-1}), k).$$

We must have $(y(k \cdot y^{-1}), k) \in K$ and therefore $(y(k \cdot y^{-1}), k) = (1, k)$, which implies that $(y, 1) \in C_G(K) \cap H$.