Exercise 1. Show that the center of a direct product is the direct product of the centers 5.1.1

$$Z(G_1 \times G_2 \times \cdots \times G_n) = Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$

Deduce that a direct product of groups is abelian if and only if each of the factors is abelian.

Solution. Suppose that $(z_1, \ldots, z_n) \in Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n)$. Then, for any $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$, we have

$$(z_1, \ldots, z_n)(g_1, \ldots, g_n) = (z_1 g_1, \ldots, z_n g_n) = (g_1 z_1, \ldots, g_n z_n) = (g_1, \ldots, g_n)(z_1, \ldots, z_n).$$

Therefore

$$Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n) \subseteq Z(G_1 \times G_2 \times \cdots \times G_n).$$

Furthermore, if $(z_1, \ldots, z_n) \in Z(G_1 \times G_2 \times \cdots \times G_n)$. Then, for any $(g_1, \ldots, g_n) \in G_1 \times \cdots \times G_n$, we have

$$(z_1 g_1, \ldots, z_n g_n) = (z_1, \ldots, z_n)(g_1, \ldots, g_n) = (g_1, \ldots, g_n)(z_1, \ldots, z_n) = (g_1 z_1, \ldots, g_n z_n).$$

Therefore

$$Z(G_1 \times G_2 \times \cdots \times G_n) \subseteq Z(G_1) \times Z(G_2) \times \cdots \times Z(G_n).$$

This establishes the result. Since any group G is abelian if and only if $Z(G) = G$ this implies the second claim.

Exercise 2. Let A and B be finite groups and let p be a prime. Prove that any Sylow p-subgroup of $A \times B$ is of the form $P \times Q$ where $P \in Syl_p(A)$ and $Q \in Syl_p(B)$. 5.1.4

Solution. Let $|G| = p^\alpha m$, where $p \not| m$. We also have that $|G| = |A| \cdot |B|$. Therefore, we have that $|A| = p_1^\alpha m_1$ and $|B| = p_2^\alpha m_2$ with $\alpha_1 + \alpha_2 = \alpha$. Clearly, there exists Sylow p-subgroups of G of the form $P \times Q$ where $P \in Syl_p(A)$ and $Q \in Syl_p(B)$. The only part to show is that no other Sylow p-subgroups of G exist.
Let $A = \{(a, 1) \mid a \in A\} \leq G$, and similarly for B. Suppose that H is a Sylow p-subgroup of G. Then $H \cap A$ is a Sylow p-subgroup of A, and $H \cap B$ is a Sylow p-subgroup of B. If both $H \cap A$ and $H \cap B$ are maximal, then H must be of the form $P \times Q$ where $P \in Syl_p(A)$ and $Q \in Syl_p(B)$. Suppose that $H \cap A$ is not maximal. Then if $P \in Syl_p(A)$, we obtain a new p-subgroup $P \times H \cap B$, such that

$$|H| \leq |H \cap A| \cdot |H \cap B| < |P \times H \cap B|,$$

which is a contradiction.

Exercise 3. Give the number of non-isomorphic abelian groups of the following orders: (a) 225, (b) 1600, (c) 1155. 5.2.1

Solution. Let $p(n)$ denote the integer partition number of n, i.e. the number of ways that n can be written as a sum of positive integers $a_1 + \cdots + a_k$ with $a_1 \leq \cdots \leq a_k$.

(a) We have that $225 = 3^2 \cdot 5^2$. Using the fundamental theorem of finitely generated abelian groups, we see that the number of abelian groups is $p(2) \cdot p(2) = 2 \cdot 2 = 4$.

(b) We have $1600 = 2^6 \cdot 5^2$. Similarly, the number of abelian groups is $p(6) \cdot p(2) = 11 \cdot 2 = 22$.

(c) We have $1155 = 3 \cdot 5 \cdot 7 \cdot 11$. There is only one abelian group on 1155 elements.

Exercise 4. Give the list of invariant factors for all abelian groups of the following orders: (a) 270, (b) 9801, (c) 165. 5.2.2

Solution. Let $p(n)$ denote the integer partition number of n, i.e. the number of ways that n can be written as a sum of positive integers $a_1 + \cdots + a_k$ with $a_1 \leq \cdots \leq a_k$.

(a) We have that $270 = 2 \cdot 3^3 \cdot 5$. The integer partitions of 3 are $3,\ 2+1,\ 1+1+1$. Therefore, we have the following groups on 270 elements (the invariant factors are the indices):

$$\mathbb{Z}_{270},\ \mathbb{Z}_{90} \times \mathbb{Z}_3,\ \mathbb{Z}_{30} \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(b) We have $9801 = 3^4 \cdot 11^2$. The integer partitions of 4 are $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$. The integer partitions of 2 are $2,\ 1+1$. Therefore, we have the following groups on 9801 elements:

$$\mathbb{Z}_{9801},\ \mathbb{Z}_{891} \times \mathbb{Z}_{11},\ \mathbb{Z}_{3267} \times \mathbb{Z}_3,\ \mathbb{Z}_{297} \times \mathbb{Z}_{33},\ \mathbb{Z}_{1089} \times \mathbb{Z}_9,$$
$$\mathbb{Z}_{99} \times \mathbb{Z}_{99}, \mathbb{Z}_{1089} \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{99} \times \mathbb{Z}_{33} \times \mathbb{Z}_3,\ \mathbb{Z}_{363} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_{33} \times \mathbb{Z}_{33} \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(c) We have $165 = 3 \cdot 5 \cdot 11$. There is only one abelian group on 165 elements: $\mathbb{Z}_{165}$.

Exercise 5. Give the list of elementary divisors for all abelian groups of the following orders: (a) 270, (b) 9801, (c) 165. 5.2.3

Solution.

(a) We have that $270 = 2 \cdot 3^3 \cdot 5$. The integer partitions of 3 are $3,\ 2+1,\ 1+1+1$. Therefore, we have the following groups on 270 elements, in terms of their elementary divisors:

$$(\mathbb{Z}_2) \times (\mathbb{Z}_{27}) \times (\mathbb{Z}_5),\ (\mathbb{Z}_2) \times (\mathbb{Z}_9 \times \mathbb{Z}_3) \times (\mathbb{Z}_5),\ (\mathbb{Z}_2) \times (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_5).$$

(b) We have $9801 = 3^4 \cdot 11^2$. The integer partitions of 4 are $4,\ 3+1,\ 2+2,\ 2+1+1,\ 1+1+1+1$. The integer partitions of 2 are $2,\ 1+1$. Therefore, we have the following groups on 9801 elements:

$$(\mathbb{Z}_{81}) \times (\mathbb{Z}_{121}),\ (\mathbb{Z}_{81}) \times (\mathbb{Z}_{11} \times \mathbb{Z}_{11})$$
$$(\mathbb{Z}_{27} \times \mathbb{Z}_3) \times (\mathbb{Z}_{121}),\ (\mathbb{Z}_{27} \times \mathbb{Z}_3) \times (\mathbb{Z}_{11} \times \mathbb{Z}_{11})$$
$$(\mathbb{Z}_9 \times \mathbb{Z}_9) \times (\mathbb{Z}_{121}),\ (\mathbb{Z}_9 \times \mathbb{Z}_9) \times (\mathbb{Z}_{11} \times \mathbb{Z}_{11})$$
$$(\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{121}),\ (\mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{11} \times \mathbb{Z}_{11})$$
$$(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{121}),\ (\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) \times (\mathbb{Z}_{11} \times \mathbb{Z}_{11}).$$

(c) We have $165 = 3 \cdot 5 \cdot 11$. There is only one abelian group on 165 elements: $(\mathbb{Z}_3) \times (\mathbb{Z}_5) \times (\mathbb{Z}_{11})$.