Let G be a finite group and let p be a prime.

Exercise 1. Prove that if $P \in Syl_p(G)$ and H is a subgroup of G containing P then $P \in Syl_p(H)$. 4.5.1

Solution. Let $|G| = p^\alpha m$ such that p does not divide m. By Lagrange, we have that the order of P divides the order of H, which in turn divides the order of G. Therefore $|H| = p^\alpha k$ such that p does not divide k. Therefore, $P \in Syl_p(H)$.

Exercise 2. Give an example to show that, in general, a Sylow p-subgroup of a subgroup of G need not be a Sylow p-subgroup of G. 4.5.1

Solution. This should happen when $H \leq G$ and $|G : H|$ is a multiple of p. For example, consider the symmetric group S_4 on 24 elements. It has a subgroup $H = \langle (1\ 2\ 3), (1\ 2) \rangle$ isomorphic to S_3 on 6 elements. Then $\langle (1\ 2) \rangle \in Syl_2(H)$, but any Sylow 2-subgroup of S_4 should have 8 elements.

Exercise 3. Use Sylow's Theorem to prove Cauchy's Theorem. 4.5.3

Solution. Let G be a group on $p^\alpha m$ elements, such that p does not divide m. We want to show that G has a subgroup of order p. Let $P \in Syl_p(G)$ and let $x \in P \setminus 1$. By Lagrange, the order of x divides p^α, and we have $|x| = p^\beta$ for $1 \leq \beta \leq \alpha$. Now, $\langle x \rangle$ is an abelian group whose order is a multiple of p, and we can apply Cauchy's Theorem for abelian groups to find an element $y \in P \leq G$ of order p.

Exercise 4. Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12}. 4.5.4

Solution. The Sylow 2-subgroups have order 4, and therefore they are either generated by an element of order 4 (which does not exist), or by two elements of order 2. It helps to see that we're looking for abelian subgroups isomorphic to V_4, and no two distinct reflections commute. Then, the possible subgroups are

$$\langle s, r^3 \rangle, \langle sr, r^3 \rangle, \langle sr, r^2 \rangle, \langle sr^3, r^3 \rangle, \langle sr^4, r^3 \rangle, \langle sr^5, r^3 \rangle.$$

The Sylow 3-subgroups have order 3, and are therefore all cyclic. Those are generated by all elements of order 3:

$$\langle r^2 \rangle = \langle r^4 \rangle.$$

Exercise 5. Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of $S_3 \times S_3$. 4.5.4

Solution. The Sylow 2-subgroups have order 4, and therefore they are either generated by an element of order 4 (which does not exist), or by two elements of order 2. It helps to see that we're looking for abelian subgroups isomorphic to V_4. This means that we are looking for subgroups of the form $\langle (x, 1), (1, y) \rangle$ where $|x| = |y| = 2$. We'll use the shorthand $ab = (a\ b)$. We have

$$\langle (12, 1), (1, 12) \rangle, \langle (12, 1), (1, 13) \rangle, \langle (12, 1), (1, 23) \rangle, \langle (13, 1), (1, 12) \rangle, \langle (13, 1), (1, 13) \rangle,$$
$$\langle (13, 1), (1, 23) \rangle, \langle (23, 1), (1, 12) \rangle, \langle (23, 1), (1, 13) \rangle, \langle (23, 1), (1, 23) \rangle.$$

By similar reasoning, the Sylow 3-subgroups have order 9 and are subgroups of the form $\langle (x, 1), (1, y) \rangle$ where $|x| = |y| = 3$.

$$\langle (123, 1), (1, 123) \rangle, \langle (123, 1), (1, 132) \rangle, \langle (132, 1), (1, 123) \rangle, \langle (132, 1), (1, 132) \rangle.$$