Exercise 1. If $\sigma \in \operatorname{Aut}(G)$ and ϕ_g is conjugation by g prove $\sigma\phi_g\sigma^{-1} = \phi_{\sigma(g)}$. Deduce that $\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G)$. 4.4.1

Solution. For any $h \in G$, we have

$$\begin{aligned}
\sigma\phi_g\sigma^{-1}(h) &= \sigma\phi_g(\sigma^{-1}(h)) \\
&= \sigma(g\sigma^{-1}(h)g^{-1}) \\
&= \sigma(g)h\sigma(g^{-1}) \\
&= \sigma(g)h\sigma(g)^{-1} \\
&= \phi_{\sigma(g)}(h),
\end{aligned}$$

by the properties of automorphisms. Therefore, the subgroup $\operatorname{Inn}(G)$ is preserved under conjugation and is therefore normal in $\operatorname{Aut}(G)$.

Exercise 2. Prove that if G is an abelian group of order pq, where p and q are distinct primes, then G is cyclic. 4.4.2

Solution. By Cauchy's Theorem, we have that G has an element, say x, of order p and an element, say y, of order q. Then $(xy)^{pq} = 1$. This implies that the order of xy divides pq. Then, the only options for $|xy|$ are $1, p, q, pq$. Since inverses have the same order, x and y are not inverses and $|xy| \neq 1$. Furthermore, $(xy)^p = y^p \neq 1$, since p does not divide q or vice versa. So $|xy| \neq p$ and similarly $|xy| \neq q$.

Exercise 3. Show that under any automorphism of D_8, r has at most 2 possible images and s has at most 4 possible images. Deduce that $|\operatorname{Aut}(D_8)| \leq 8$. 4.4.3

Solution. Since order of elements is preserved under automorphisms, the element r must map to an element of order 4, of which there are only two: r and r^3. Similarly, s must map to an element of order 2, of which there are five: r^2, s, sr, sr^2, sr^3. However, we have $r^2 \in Z(G)$ and $s \notin Z(G)$, so therefore s has at most 4 possible images. Since r and s generate D_8, their mapping determines the full automorphism. Therefore, $|\operatorname{Aut}(D_8)| \leq 2 \cdot 4 = 8$.