11/2

Projections to some $V \subseteq \mathbb{R}^n$.

If we have an orthonormal basis for V:

$\langle \hat{u}_1 \ldots \hat{u}_p \rangle$ Then we can write this projection as

$$\text{proj}_V \vec{x} = (\hat{u}_1 \cdot \vec{x})\hat{u}_1 + \ldots + (\hat{u}_p \cdot \vec{x})\hat{u}_p .$$

and projection matrix $A = QQ^T$

$$Q = \begin{pmatrix} \uparrow & & \uparrow \\ \hat{u}_1 & \ldots & \hat{u}_p \\ \downarrow & & \downarrow \end{pmatrix}.$$

Finding orthonormal bases: Gram-Schmidt Algorithm.

Suppose we have any basis $\langle \vec{v}_1 \ldots \vec{v}_p \rangle$ for V:

we will turn this in to an orthonormal one.

- let $\hat{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$.

- let $\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - \text{proj}_W \vec{v}_2$ $\quad W = [\hat{u}_1]$.

  and $\hat{u}_2 = \dfrac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|}$.

- let $\vec{v}_3^{\perp} = \vec{v}_3 - \text{proj}_W \vec{v}_3$ $\quad W = [\hat{u}_1, \hat{u}_2]$

(Span)

and $\hat{u}_3 = \dfrac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|}$.

$\vdots$

- let $\vec{v}_i^{\perp} = \vec{v}_i - \text{proj}_W \vec{v}_i \qquad W = [\hat{u}_1 \dots \hat{u}_{i-1}]$

$\hat{u}_i = \dfrac{\vec{v}_i^{\perp}}{\|v_i^{\perp}\|}$.

$\text{proj}_W \vec{v}_i = (\hat{u}_1 \cdot \vec{v}_i)\,\hat{u}_1 + \dots + (\hat{u}_{i-1} \cdot \vec{v}_i)\,\hat{u}_{i-1}$.

If we have any basis for V, we can turn this in to an orthonormal one: $\langle \hat{u}_1, \dots, \hat{u}_p \rangle$

We can then extend this to a basis for $\mathbb{R}^n$:

$$\langle \hat{u}_1 \dots \hat{u}_p, \vec{v}_{p+1} \dots \vec{v}_n \rangle$$

We can then use Gram-Schmidt again to find an orthonormal basis for all of $\mathbb{R}^n$:

$$\langle \underbrace{\hat{u}_1 \dots \hat{u}_p}_{\text{basis for } V} \; \underbrace{\hat{u}_{p+1} \dots \hat{u}_n}_{\text{basis for } V^{\perp}} \rangle.$$

Rank-nullity Thm

Let A be some $m \times n$ matrix.

Then $\text{rank}(A) = \text{rank}(A^T)$. (since rank is the dimension of the column span or row span; those are equal).

If $\vec{v} \in \ker(A)$

Then $A\vec{v} = \vec{0}$

$$A\vec{v} = \begin{pmatrix} \leftarrow \vec{r}_1 \rightarrow \\ \vdots \\ \leftarrow \vec{r}_n \rightarrow \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{v}$ is perpendicular to the row span of A.

If V is the row span of A then

$\ker(A) = V^{\perp}$.

$\text{rank}(A) + \text{nul}(A) = \dim(V) + \dim(V^{\perp}) = n$.

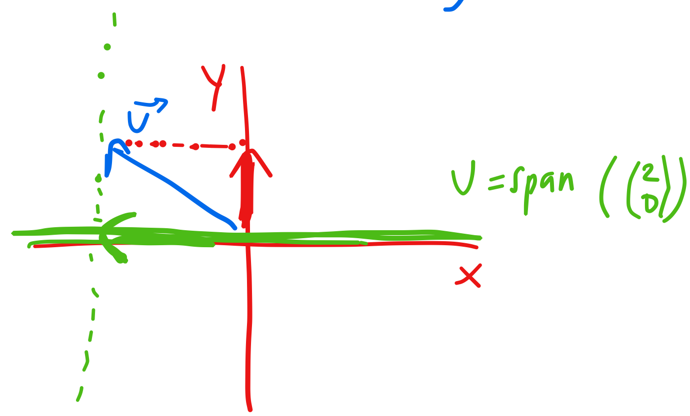
Example: $A = \begin{pmatrix} 2 & 0 \end{pmatrix}$ $\qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2x$

$V =$ row span of A

$V \subseteq \mathbb{R}^2$ $\qquad \mathbb{R}^2 \xrightarrow{A} \mathbb{R}$

See that $\text{im}(A) = \mathbb{R}$

and $\dim(V) = \dim(\mathbb{R})$
$= \dim(\text{im}\, A)$.



<u>Application:</u> If $A\vec{v} = \vec{w}$ doesn't have a solution, can we find the "closest" answer?

$\Rightarrow$ $\vec{w}$ is not in $\text{im}(A) =$ span of the columns.
$=$ range space

Then there is a vector in $\text{im}(A)$ that is closest to $\vec{w}$: namely the projection $\text{proj}_{\text{im}(A)} \vec{w} = \vec{w}^{\parallel}$

Then $A\vec{v}^* = \vec{w}^{\parallel}$ does have a solution!

We call $\vec{v}^*$ a least squares solution, since it minimizes the length of the error $\vec{w} - A\vec{v}^*$.

The error vector $\vec{w} - A\vec{v}^*$ is in the $(\text{im}\, A)^{\perp}$

$\vec{w} - \vec{w}^{\parallel} = \vec{w}^{\perp}$ to $\text{im}\, A$.

$=$ column span of A.

$\Rightarrow$ A^T ...

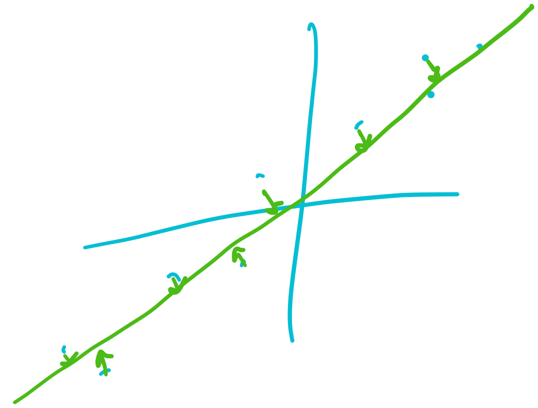
$A^T \vec{w}^{\perp} = \vec{0}$

$A^T(\vec{w} - \vec{w}^{\parallel})$

$= A^T(\vec{w} - A\vec{v}^*)$

$= \vec{0}$

Each entry here is the dot product of $\vec{w}^{\perp}$ with a row of A^T = column of A.



$A^T\vec{w} - A^TA\vec{v}^* = \vec{0}$

$$A^TA\vec{v}^* = A^T\vec{w}.$$

Suppose we want to solve $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$.

Has no solution. $\operatorname{rref}\left(\begin{array}{cc|c}1 & 2 & 1\\ 2 & 4 & 1\end{array}\right) = \left(\begin{array}{cc|c}1 & 2 & 1\\ 0 & 0 & 1\end{array}\right)$

~~The~~ A least squares approximate solution is any solution to

$A = \begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix} = A^T$

$\vec{w} = \begin{pmatrix}1\\1\end{pmatrix}$

$$A^TA\vec{v}^* = A^T\vec{w}$$

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\vec{v}^* = \begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$$

$\begin{pmatrix}5 & 10\\ \end{pmatrix}\vec{v}^* = \begin{pmatrix}3\\ \end{pmatrix}$ There are infinitely

$\begin{pmatrix} 10 & 20 \end{pmatrix}$ $\begin{pmatrix} 6 \end{pmatrix}$

many solutions for $\vec{v}^*$

but only 1 for $A\vec{v}^*$ $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} (A\vec{v}^*) = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$$= \begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix}$$

Then the error $\| \vec{w} - A\vec{v}^* \| = \frac{1}{\sqrt{5}}$.

Nov. 4

Consider a square $n \times n$ matrix A.

Then this represents a linear transformation of $\mathbb{R}^n$.

If A has rank n (full rank) then it represents an automorphism on $\mathbb{R}^n$ and should have a 2-sided inverse: A^{-1}.

$$A = \begin{pmatrix} a & c \cdot a \\ b & c \cdot b \end{pmatrix} \iff \begin{array}{l} a \times cb = b \times ca \\ a \times cb - b \times ca = 0. \end{array}$$

Definition: For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant is given by $ad - bc$.

↳ also written as $|A|$ or $\det A$

For a 3×3 matrix

$$\det\begin{pmatrix} a & b & c \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = +a \cdot \det\begin{pmatrix} a & b & c \\ \cdot & \boxed{\cdot \quad \cdot} \\ \cdot & \boxed{\cdot \quad \cdot} \end{pmatrix}$$

$$-b \cdot \det\begin{pmatrix} a & b & c \\ \boxed{\cdot} & \cdot & \boxed{\cdot} \\ \boxed{\cdot} & \cdot & \boxed{\cdot} \end{pmatrix} + c \cdot \det\begin{pmatrix} a & b & c \\ \boxed{\cdot \quad \cdot} & \cdot \\ \boxed{\cdot \quad \cdot} & \cdot \end{pmatrix}$$

Example

$$\det\begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 1 \\ 3 & 1 & 0 \end{pmatrix} = 1 \cdot \det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$-2 \det\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix}$$

$$= 1 \cdot (0 \cdot 0 - 1 \cdot 1) - 2 \cdot (-1 \cdot 0 - 1 \cdot 3) + 0 \cdot (-1 \cdot 1 - 0 \cdot 3)$$

$$= -1 + 6 = 5.$$

Eigenvectors & Eigenvalues

$\in \mathbb{R}^n$

A nonzero vector $\vec{v}$ is called an eigenvector of an $n \times n$ matrix A if $A\vec{v} = \lambda \vec{v}$

Then λ is the eigenvalue associated to $\vec{v}$. $\lambda \in \mathbb{R}$.

Suppose that $\vec{v}$ is an eigenvector of A, then is $c\vec{v}$ an eigenvector?

(A is a linear map)

$$A(c\vec{v}) = c \cdot (A\vec{v}) = c \cdot \lambda \cdot \vec{v} = \lambda \cdot (c\vec{v})$$

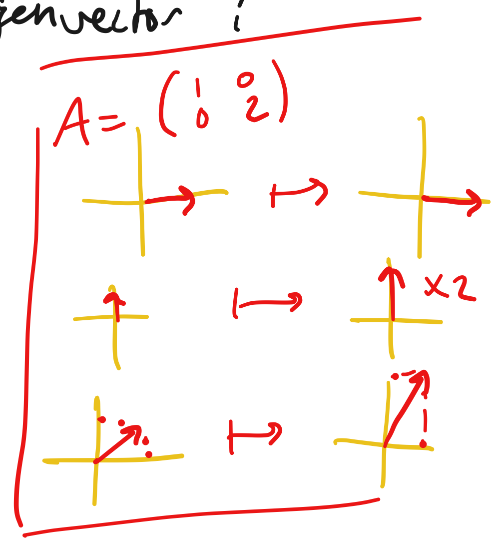
$\Rightarrow$ $c\vec{v}$ is also an eigenvector.

Suppose that $\vec{v}$ and $\vec{w}$ are both eigenvectors of A, then is $\vec{v} + \vec{w}$ an eigenvector?

$$A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = \lambda_1 \vec{v} + \lambda_2 \vec{w}$$

where λ_1 is the eigenvalue of $\vec{v}$ and λ_2 the eigenvalue of $\vec{w}$.

$\Rightarrow$ $A(\vec{v} + \vec{w}) = \lambda(\vec{v} + \vec{w})$ if $\lambda_1 = \lambda_2 = \lambda$.

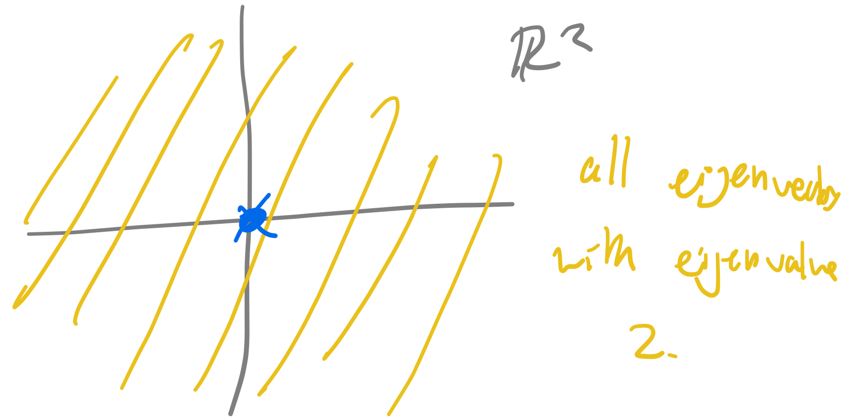


Example : Projection in $\mathbb{R}^2$ to the x-axis.

eigenvectors with eigenvalue 0

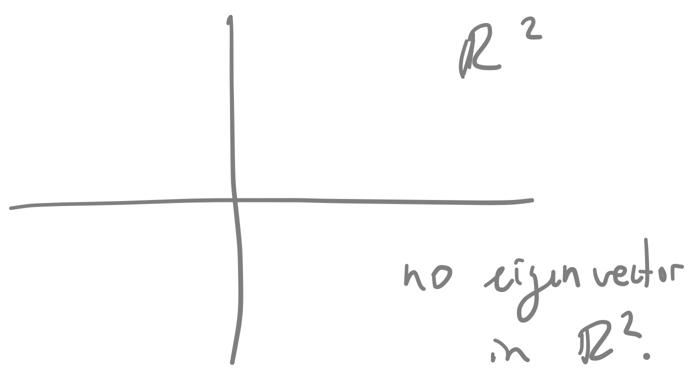
$A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$

Scaling: $A=\begin{pmatrix}2&0\\0&2\end{pmatrix}$



Rotation:

$A=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$



Finding eigenvalues:

$$A\vec{v}=\lambda\vec{v}$$

$$A\vec{v}-\lambda\vec{v}=\vec{0}.$$

$$A\vec{v}-\lambda I_n\vec{v}=\vec{0}$$

$$(A-\lambda I_n)\vec{v}=\vec{0}.$$

$$\boxed{\begin{aligned}A\vec{v}-B\vec{v}&=0\\(A-B)\vec{v}&=0.\end{aligned}}$$

($\vec{v}$ is nonzero).

Then the matrix $A-\lambda I_n$ is not invertible.

$$\Rightarrow \quad \underbrace{\det(A-\lambda I_n)} = 0.$$

because of how determinants are defined, this is always a polynomial in λ of degree n.

$$\lambda \cdot I_n = \lambda \cdot \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

$$A - \lambda I_n = \begin{pmatrix} a_{11}-\lambda & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} \cdots a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn-1} \quad a_{nn}-\lambda \end{pmatrix}$$

Example.

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

$$A - \lambda I_2 = \begin{pmatrix} 0-\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix} \qquad \det(A-\lambda I_n) = 0$$

$$(0-\lambda)(-3-\lambda) - (1 \cdot -2) = \lambda^2 + 3\lambda + 2$$

Maybe you can tell that this determinant of $A-\lambda I_n$ is always a polynomial of order n.

We need $\lambda^2 + 3\lambda + 2 = 0$.

$(\lambda+1)(\lambda+2) = 0.$

Our eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$.

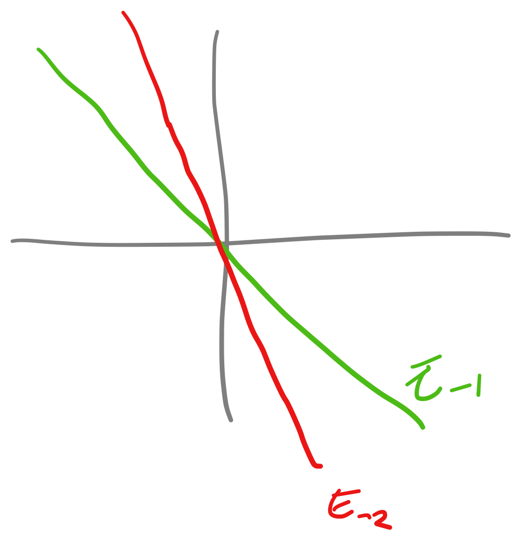
The eigenvectors of -1 live in the eigenspace

$$E_{-1} = \ker \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} = \operatorname{span}\left(\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

$\boxed{\text{solutions to } \left(\begin{array}{cc|c} 1 & 1 & 0 \\ -2 & -2 & 0 \end{array}\right).}$

$$E_{-2} = \ker \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}$$

$$= \operatorname{span}\left(\begin{pmatrix} 1 \\ -2 \end{pmatrix} \right).$$



Multiplicities:

We call $\det(A - \lambda I_n)$ the characteristic polynomial. If this can be written

as $(\lambda_1 - \lambda)^k \cdot h(\lambda)$ with $h(\lambda_1) \neq 0$.

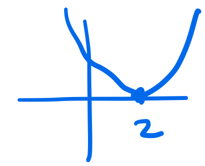
We call k the algebraic multiplicity of λ_1

notation: $\text{almu}(\lambda_i) = k$.

We can always write $f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$.

where $f_A(\lambda_i) = 0$.

Example: $(1-\lambda)(2-\lambda)$ or $(2-\lambda)^2$



alg. mult of 1 is 1

" " " 2 is 1

alg. mult. of 2 is 2.

$(2-\lambda)^4 (i-\lambda)^6$ the eigenvalue 2 has alg. mult. 4 and i has alg. mult 6.

$\text{almu}(2) = 4$

$\text{almu}(i) = 6$.

We call $\dim E_\lambda$ the geometric multiplicity of λ. $\text{gemu}(\lambda)$.

Fact: $\text{gemu}(\lambda) \leq \text{almu}(\lambda)$.

Example: Consider $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $A_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$$f_{A_1}(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2$$

A_1 has eigenvalue 1. (with almu 2)

$$E_1 = \ker\begin{pmatrix}1-1 & 0\\ 0 & 1-1\end{pmatrix} = \ker\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = \mathbb{R}^2$$

$$\text{gemu}(1) = \dim E_1 = 2.$$

$$f_{A_2}(\lambda) = \det\begin{pmatrix}1-\lambda & 1\\ 0 & 1-\lambda\end{pmatrix} = (1-\lambda)^2$$

A_2 has eigenvalue 1 with almu = 2.

$$E_1 = \ker\begin{pmatrix}1-1 & 1\\ 0 & 1-1\end{pmatrix} = \ker\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \left[\begin{pmatrix}1\\0\end{pmatrix}\right]$$

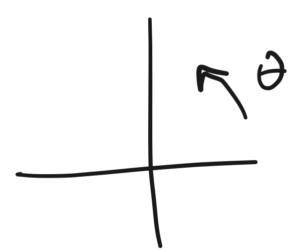
$\dim E_1 = 1 = \text{gemu}(1)$

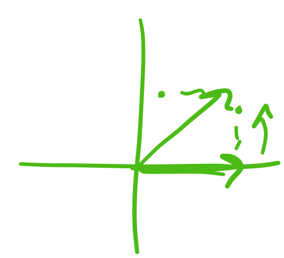
$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \iff y = 0.$

Rotations in 2D

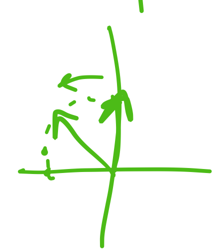
$\theta = 45°$

A rotation over θ





has matrix $\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$



suppose $\sin\theta \neq 0$. and let $a = \cos\theta$
$b = \sin\theta$.

$A = \begin{pmatrix}a & -b\\ b & a\end{pmatrix}$ has $f_A(\lambda) = (a-\lambda)(a-\lambda) + b^2$

$$= \lambda^2 - 2a\lambda + a^2 + b^2$$

$$= (a+bi-\lambda)(a-bi-\lambda)$$

A has eigenvalues $\lambda_1 = a+bi$ $\lambda_2 = a-bi$.

$$E_{a+bi} = \ker \begin{pmatrix} a-(a+bi) & -b \\ b & a-(a+bi) \end{pmatrix} = \ker \begin{pmatrix} -bi & -b \\ b & -bi \end{pmatrix}$$

$$= \left[\begin{pmatrix} i \\ 1 \end{pmatrix} \right] \qquad i \begin{pmatrix} -bi \\ b \end{pmatrix} = \begin{pmatrix} b \\ bi \end{pmatrix}$$

$$E_{a-bi} = \ker \begin{pmatrix} a-(a-bi) & -b \\ b & a-(a-bi) \end{pmatrix} = \ker \begin{pmatrix} bi & -b \\ b & bi \end{pmatrix}$$

$$= \left[\begin{pmatrix} i \\ -1 \end{pmatrix} \right]. \qquad i \begin{pmatrix} bi \\ b \end{pmatrix} = \begin{pmatrix} -b \\ bi \end{pmatrix}$$

Rotations in 3D?

$$A = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$$

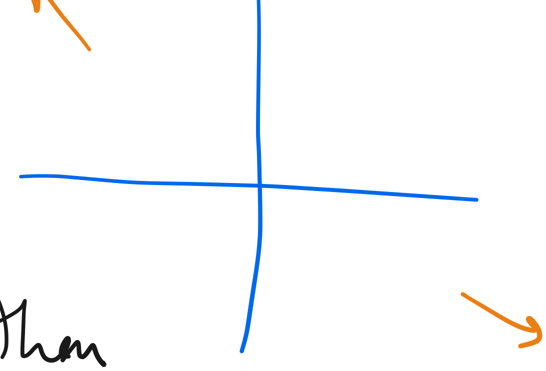
Let A be a 3×3 matrix.

Then what is $f_A(\lambda) = -\lambda^3 + .. \lambda^2 -- \lambda .. c$

$f_A \to -\infty$ as $\lambda \to \infty$

$f_A \to -\infty$ as $\lambda \to \infty$

$f_A \to \infty$ as $\lambda \to -\infty$



By the intermediate value thm (and some more details) we have at least 1 real eigenvalue.

11/16. If A (n×n matrix) has a real eigenvalue λ, then the matrix $A - \lambda I_n$ has real entries and non zero kernel $\Rightarrow$ kernel is a subspace of $\mathbb{R}^n$ $\Rightarrow$ then λ has real eigenvectors.

However, if $\vec{v}$ is an eigenvector then for example $i\vec{v}$ is also an eigenvector.

$\Rightarrow$
- Real eigenvalues imply the existence of real eigenvectors (but may also have non-real ones)
- Not-real eigenvalues have not-real eigenvectors.

An $n \times n$ matrix A is invertible

$\iff \ker A = \{\vec{0}\} \iff \operatorname{rank} A = n.$

Then there exists an inverse matrix A^{-1} such that $AA^{-1} = I_n$. $(A^{-1})^{-1} = A$.
$= A^{-1}A$

In general, we can use rref to find the inverse of a matrix:

$$\left(A \mid I_n\right) \xrightarrow{\text{rref}} \left(I_n \mid A^{-1}\right).$$

Example $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $a = \cos\theta$
$b = \sin\theta$

$$\left(\begin{array}{cc|cc} a & -b & 1 & 0 \\ b & a & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & -b/a & \frac{1}{a} & 0 \\ 0 & a+b^2/a & -b & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & -b/a & 1/a & 0 \\ 0 & 1 & \frac{-b}{a+b^2/a} & 1/a+b^2/a \end{array}\right)$$

$$\to \quad - \quad A^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Easier: $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right). \Rightarrow A^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^{-1}A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For any basis $B = \langle \vec{\beta}_1, \dots \vec{\beta}_n \rangle$ of $\mathbb{R}^n$

we have that for any $\vec{v} \in \mathbb{R}^n$

$$\vec{v} = c_1 \cdot \vec{\beta}_1 + \dots + c_n \vec{\beta}_n$$

We write $\mathrm{Rep}_B \vec{v} = [\vec{v}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}_B$

if $S = \begin{pmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{\beta}_1 & \vec{\beta}_2 & \dots & \vec{\beta}_n \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}$ then

$[\vec{v}]_E = \vec{v} = S[\vec{v}]_B.$ S is an $n \times n$

$I_n \vec{v} =$ <u>invertible</u> matrix.

$\rightarrow$ rank n.

$$\vec{v} = S[\vec{v}]_{\mathcal{B}}$$
$$S^{-1}\vec{v} = S^{-1}S[\vec{v}]_{\mathcal{B}}$$
$$S^{-1}\vec{v} = [\vec{v}]_{\mathcal{B}}.$$

$\Rightarrow$ Change of basis is itself a linear transformation.

Suppose we have a linear transformation of $\mathbb{R}^n$ given by A.

Suppose I want to find the matrix of this transformation w.r.t. $\mathcal{B}$-basis.

We know that

$\hookrightarrow$ we want

$$[A\vec{v}]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}.$$

$$B = \begin{pmatrix} \uparrow & & \uparrow \\ [A\vec{\beta}_1]_{\mathcal{B}} & \cdots & [A\vec{\beta}_n]_{\mathcal{B}} \\ \downarrow & & \downarrow \end{pmatrix}$$

$\hookrightarrow$ we call this the B-matrix of f.

If f is represented by A then this is $\mathrm{Rep}_{\mathcal{B},\mathcal{B}} f$.

Suppose $A\vec{v} = \vec{w} = A\,S[\vec{v}]_{\mathcal{B}}$

$$[\vec{w}]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}$$

$\vec{w} = S[\vec{w}]_{\mathcal{B}}$

$S B$ ...

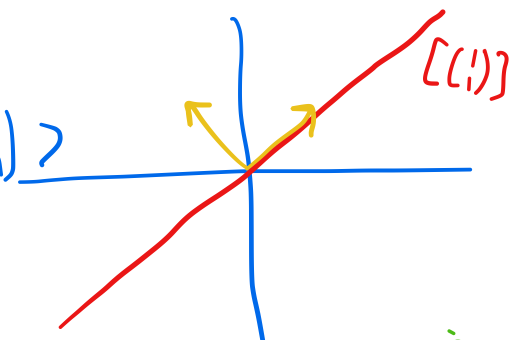
$= SB[\vec{v}]_B$

$\Rightarrow \quad AS = SB \qquad B = S^{-1}AS.$

$S^{-1}AS = \cancel{S^{-1}}SB \qquad A = SBS^{-1}.$

Nov 18

Example find the matrix A (standard basis) of the projection to the line $L = [\binom{1}{1}]$ in $\mathbb{R}^2$.



If we work w.r.t. $\mathcal{B} = \langle \binom{1}{1}, \binom{1}{-1} \rangle$

Then the B matrix is

$$B = \left(f\left(\binom{1}{1}\right)_{\mathcal{B}} \quad f\left(\binom{1}{-1}\right)_{\mathcal{B}} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\mathcal{B},\mathcal{B}} \longrightarrow \mathrm{Rep}_{\mathcal{B},\mathcal{B}}\,\mathrm{proj}_L$$

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad A = SBS^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1}.$$

Exercise: compare this A to what you would get using Gram-schmidt $(A = QQ^T)$.

In the Homework, you showed that for a 2×2 matrix A, we have

$$f_A(\lambda) = \lambda^2 - tr(A)\lambda + det(A)$$

This pattern applies more generally: for an $n \times n$ matrix A, we have

$$f_A(\lambda) = (-\lambda)^n + tr(A)(-\lambda)^{n-1} + \overbrace{\cdots}^{\text{something else}} + det(A).$$

(this follows in a similar way from the definition of the determinant of $A - \lambda I_n$).

We also know that if $\lambda_1 \dots \lambda_n$ are the eigenvalues of A (with multiplicities) then

$$f_A(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

$(a-x)(b-x) = x^2 + (a+b)(-x) + ab.$

$$= (-\lambda)^n + (\lambda_1 + \lambda_2 + \dots \lambda_n)(-\lambda)^{n-1} + \dots\dots + \lambda_1\lambda_2\dots\lambda_n.$$

since coefficients are equal in both cases

$$\Rightarrow \quad tr(A) = \sum_{i=1}^{n} \lambda_i \qquad det(A) = \prod_{i=1}^{n} \lambda_i .$$

<u>Similar matrices</u> For two $n \times n$ matrices A, B we say that

$A \sim B$ (A is similar to B) if there exists an invertible $n \times n$ matrix S such that $A = SBS^{-1}$. $\quad AS = SB$.

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map.
such that $A = \mathrm{Rep}_{E,E}\, f \qquad B = \mathrm{Rep}_{\mathcal{B},\mathcal{B}}\, f$

$$f(\vec{v}) = A\vec{v}$$

$$\begin{tikzcd}
\vec{v} \arrow[r, "A"] \arrow[d, shift left, "S^{-1}"] & f(\vec{v}) \arrow[d, shift left, "S^{-1}"] \\
{[\vec{v}]_{\mathcal{B}}} \arrow[r, "B"] \arrow[u, shift left, "S"] & {[f(\vec{v})]_{\mathcal{B}}} \arrow[u, shift left, "S"]
\end{tikzcd}$$

$$[f(\vec{v})]_{\mathcal{B}} = B[\vec{v}]_{\mathcal{B}}$$

$$\vec{v} = S[\vec{v}]_{\mathcal{B}}$$

$$S^{-1}\vec{v} = [\vec{v}]_{\mathcal{B}}$$

$$A = SBS^{-1}$$
$$B = S^{-1}AS$$

This relation $A \sim B \iff A = SBS^{-1}$ for some invertible S.

is an equivalence relation.

Matrices are similar iff they have the same

Jordan canonical form / Smith

(analogous to the rref)

Diagonalizable matrices

Diagonal matrix: $\begin{pmatrix} \times & & & & & 0 \\ & \times & & & & \\ & & \times & & & \\ & & & \times & & \\ & & & & \times & \\ 0 & & & & & \times \end{pmatrix}$ $n \times n$

All nonzero entries are on the diagonal.

Example: What are the eigenvalues of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We see
$$A\begin{pmatrix}1\\0\\0\end{pmatrix} = 1\begin{pmatrix}1\\0\\0\end{pmatrix}$$
$$A\begin{pmatrix}0\\1\\0\end{pmatrix} = 2\begin{pmatrix}0\\1\\0\end{pmatrix}$$
$$A\begin{pmatrix}0\\0\\1\end{pmatrix} = 0\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\Rightarrow$ Eigenvalues are the diagonal entries.

Also true if A is not equal, but similar to a diagonal matrix.

We say A is diagonalizable if A is similar to a diagonal matrix

$\iff$ A has an ei

A has an eigenbasis:

a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

$\Leftrightarrow$ dimensions of eigenspaces add up to n.

Similar matrices have the same

- Characteristic polynomial $\rightarrow$ $\left\{ \begin{array}{l} \text{- eigenvalues} \\ \text{- algebraic mult} \\ \text{- trace} \\ \text{- determinant} \\ \text{- invertibility.} \end{array} \right.$

- geometric multiplicities $\rightarrow$ $\left\{ \begin{array}{l} \text{rank} \\ \text{nullity} \\ \text{diagonalizability.} \end{array} \right.$

- ~~eigenvectors~~
- ~~kernel~~
- ~~image~~

11/30

Two $n \times n$ matrices A and B are similar iff they have the same Jordan Normal Form.

(analogous to rref)

A is diagonalizable iff its JNF is

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where $\lambda_1 \ldots \lambda_n$ are the eigenvalues (with multiplicities). (ordered by absolute value).

Roughly speaking: the JNF is as close to such a diagonal matrix as possible in general.

Discrete Dynamical Systems

Takes the form $\vec{x}(t)$ $\vec{x}(0), \vec{x}(1), \vec{x}(2) \ldots$

$$\begin{aligned} \vec{x}(t) &= A\vec{x}(t-1) \\ &= A(A\vec{x}(t-2)) \ldots \\ &= A^t \vec{x}(0). \end{aligned}$$

where A is some $n \times n$ matrix

(this is not A^T).

Suppose that A is diagonalizable.

Then A has eigenvalues $\lambda_1 \ldots \lambda_n$

(not necessarily distinct)

and corresponding eigenbasis $\langle \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \rangle =: \mathcal{B}$.

Then we can write our

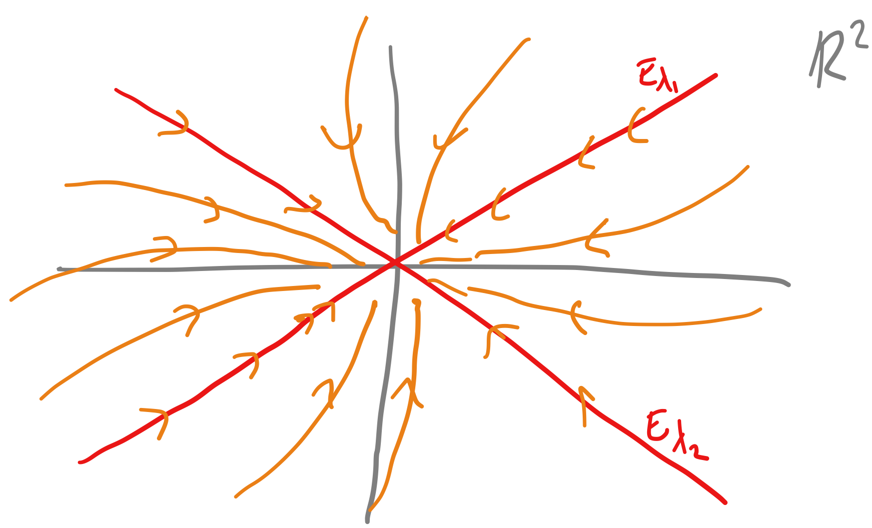
... initial state

$$\vec{x}(0) = \vec{x}_0 = c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \left[\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}\right]_B$$

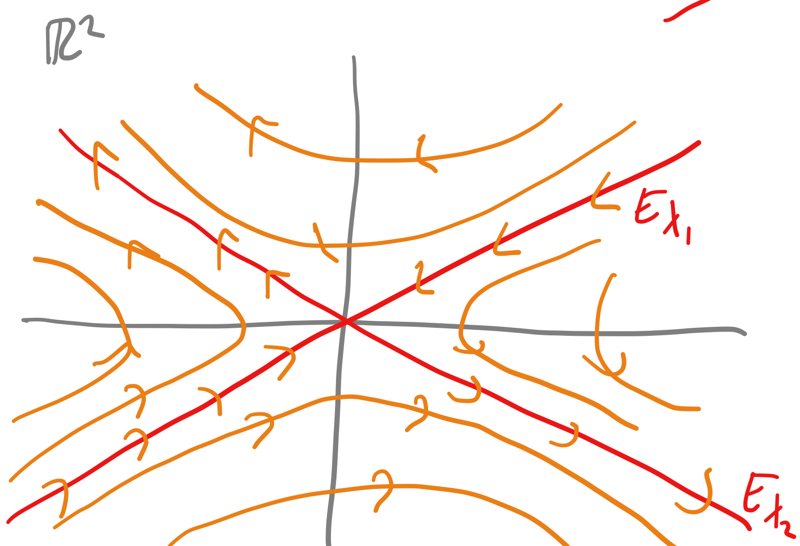
$$\vec{x}(t) = A^t\vec{x}_0 = A^t(c_1\vec{v}_1 + \dots + c_n\vec{v}_n)$$
$$= c_1(A^t\vec{v}_1) + \dots + c_n(A^t\vec{v}_n)$$
$$= c_1\lambda_1^t\vec{v}_1 + \dots + c_n\lambda_n^t\vec{v}_n = \sum_{i=1}^{n} c_i\lambda_i^t\vec{v}_i.$$

Consider 2x2 case.

Phase portraits.



$0 < \lambda_1 \leq \lambda_2 < 1$



$0 < \lambda_1 < 1 < \lambda_2$

We'll consider $\lambda = 1$ later.

Example: Fibonnacci as a discrete ...

discrete dynamical system.

$\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$ want $\vec{x}(t) = \begin{pmatrix} F_t \\ F_{t+1} \end{pmatrix}$.

let $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ $A \begin{pmatrix} F_i \\ F_{i+1} \end{pmatrix} = \begin{pmatrix} F_{i+1} \\ F_{i+2} \end{pmatrix}$.

$$f_A(\lambda) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = -\lambda(1-\lambda) - 1$$
$$= \lambda^2 - \lambda - 1. \quad = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$\varphi = \frac{1+\sqrt{5}}{2}$ $\psi = \frac{1-\sqrt{5}}{2}$.

$\varphi > 1$ $-1 < \psi < 0$.

Eigenvectors: $\vec{v}_1 = \begin{pmatrix} 1 \\ \varphi \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ \psi \end{pmatrix}$.

As t increases $\vec{x}(t)$ looks more and more like a multiple of $\vec{v}_1$

(component along $\vec{v}_2$ vanishes)

$$\vec{x}(t) = c_1 \varphi^t \vec{v}_1 + c_2 \underbrace{\psi^t}_{\to 0 \text{ as } t \to \infty} \vec{v}_2$$

Now we see that $F_t / F_{t-1} \to 4$ as $t \to \infty$.

<u>Example</u> Simple predator-prey model.

Suppose we have two animals foxes and mice, and we want to model population sizes over time.

F_t : # foxes at time t

M_t : # mice " " "

$$F_{t+1} = .4 F_t + .5 M_t$$
$$M_{t+1} = -.2 F_t + 1.2 M_t$$

$$\vec{x}_0 = \begin{pmatrix} F_0 \\ M_0 \end{pmatrix}$$

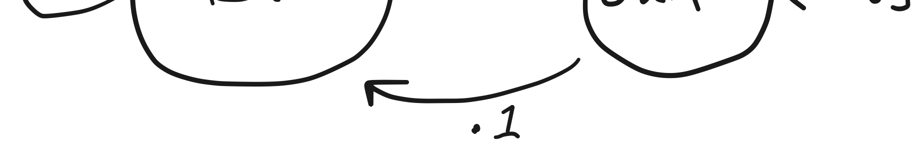
Then $\vec{x}(t) = A^t \vec{x}_0$ $\quad A = \begin{pmatrix} .4 & .5 \\ -.2 & 1.2 \end{pmatrix}$.

We can find $\lambda_1 = 1.045$ $\quad \lambda_2 = .55$

and eigenvectors $\vec{v}_1 \approx \begin{pmatrix} .6 \\ .8 \end{pmatrix}$ $\quad \vec{v}_2 \approx \begin{pmatrix} .9 \\ .3 \end{pmatrix}$.

<u>Dec 2</u>

Weather system



In this system if it rains at time 0,
then $\mathbb{P}$(sunny at time 1) = 20%
$\mathbb{P}$(rain " " ") = 80%.

We let $\vec{x}_t$ be a <u>distribution vector</u>: reflects the probability of each state at time t.
→ vector with all entries adding up to 1, and entries between 0 and 1.

$\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $\vec{x}_1 = \begin{pmatrix} .8 \\ .2 \end{pmatrix}$.

What about $\vec{x}_2$?

$$\mathbb{P}(\text{Rain at } t=2) = \mathbb{P}(\text{rain at } t=1) \cdot \mathbb{P}(\text{rain at } t=2 \mid \text{rain at } t=1)$$

$$+ \, \mathbb{P}(\text{sun at } t=1) \cdot \mathbb{P}(\text{rain at } t=2 \mid \text{sun at } t=1).$$

$$= .8 \cdot .8 + .2 \cdot .1 = (\vec{x}_2)_1$$

$$\vec{x}_2 = \begin{pmatrix} .8 & .1 \\ .2 & .9 \end{pmatrix} \vec{x}_1$$

In general: $\vec{x}_{t+1} = \begin{pmatrix} .8 & .1 \\ .2 & .9 \end{pmatrix} \vec{x}_t$.

→ This is a transition matrix: entries between 0 and 1 <u>and</u> all columns add up to 1.

Claim: if $\vec{x}$ is a distribution vector and A a transition matrix, then $A\vec{x}$ is a distribution vector.

Proof: $A\vec{x} = \begin{pmatrix} \uparrow & & \uparrow \\ \vec{v}_1 & \cdots & \vec{v}_n \\ \downarrow & & \downarrow \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{v}_1 + \dots + x_n\vec{v}_n$.

Each vector $x_i\vec{v}_i$ has total sum $= x_i$.

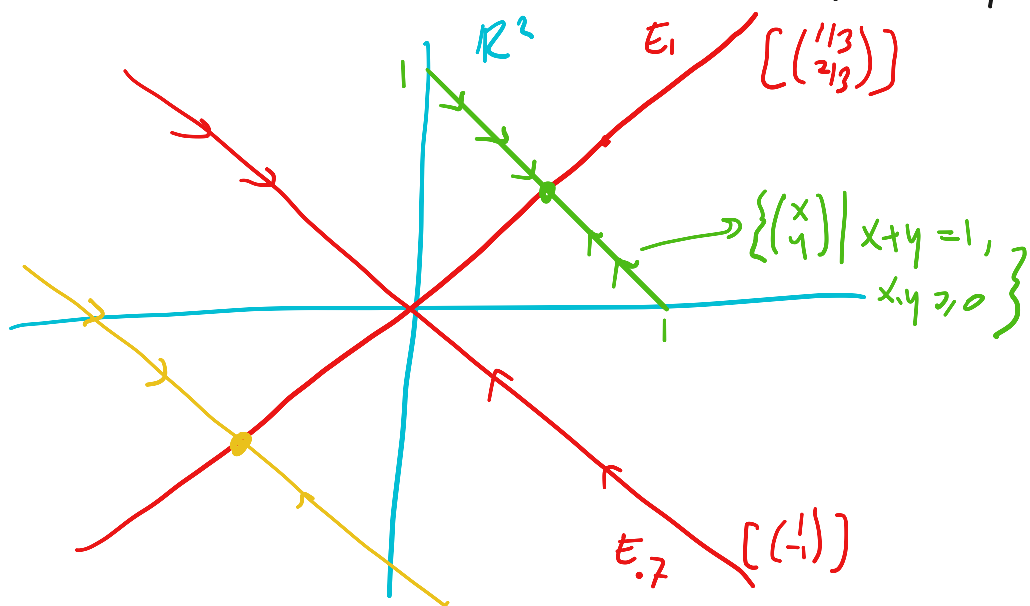
$\Rightarrow$ Total sum $A\vec{x} = x_1 + \dots + x_n = 1$. $\square$

If A has $\underbrace{}_{}^{\text{a distribution}}$ eigenvector $\vec{v}$ with eigenvalue 1, then this is an equilibrium distribution:

$A\vec{v} = \vec{v}$.

In this case: A has eigenvectors $\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalues 1 and .7 respectively.

Phase portrait

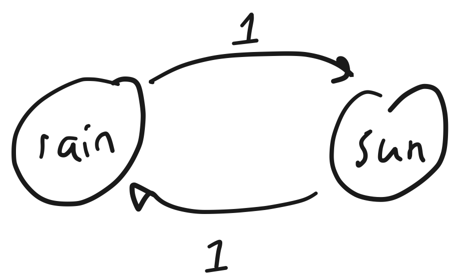


We see that for any distribution $\vec{x}_0$ we have $\vec{x}_t \to \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$ as $t \to \infty$.

→ we call this the equilibrium distribution.

This does not always work.

For example, what if we had the system

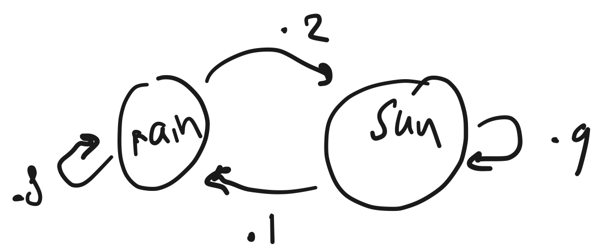


This has transition matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

We have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

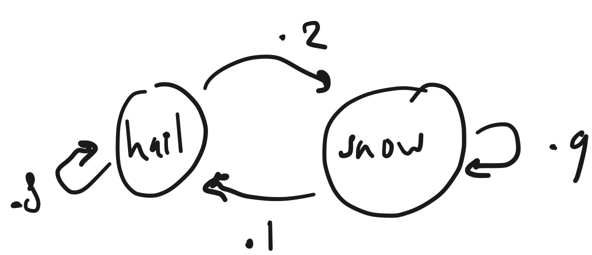
If $\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then we get $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ldots$

Another example:



Then if $\vec{x}_0 = \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{l} \text{rain} \\ \text{sun} \\ \text{hail} \\ \text{snow} \end{array}$

then $\vec{x}_t \to \begin{pmatrix} 1/3 \\ 2/3 \\ 0 \\ 0 \end{pmatrix}$



but if $\vec{x}_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ then $\vec{x}_t \to \begin{pmatrix} 0 \\ 0 \\ 1/3 \\ 2/3 \end{pmatrix}$.

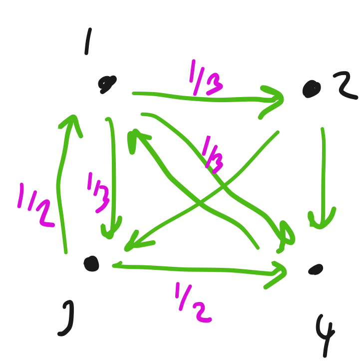
A transition matrix is <u>positive</u> if all entries are >0. We say that a transition matrix A is <u>regular</u> if A^m is positive for some integer m.

<u>Thm</u> If A is a regular transition matrix, then $\exists$ an equilibrium distribution vector $\vec{x}_{eq}$ such that $A\vec{x}_{eq} = \vec{x}_{eq}$ and

$A^t \vec{x}_0 = \vec{x}_t \longrightarrow \vec{x}_{eq}$ for <u>any</u> initial distribution $\vec{x}_0$.

Suppose we have a system of websites and links.

A user is equally likely to follow any link from where they currently are.



Transition matrix $A = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 1 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 0 \\ 1/3 & 1/2 & 1/2 & 0 \end{pmatrix}$

In a transition matrix A the entry ij indicates probability $j \to i$.

A is not positive, but A^4 is. $\Rightarrow$ A is regular.

If we ask computer for eigenvalues/eigenvectors, we will find a $\vec{x}_{eq}$ (with eigenvalue 1).

In this case $\vec{x}_{eq} \sim \begin{pmatrix} .4 \\ .1 \\ .2 \\ .3 \end{pmatrix}$.

This means that users spend $\sim 40\%$ of the time on website 1, or at any time $\sim 40\%$ of users are on website 1.