Abstraction and grounding

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Abstract

The idea that some objects are metaphysically "cheap" has wide appeal. An influential version of the idea builds on abstractionist views in the philosophy of mathematics, on which numbers and other mathematical objects are abstracted from other phenomena. For example, Hume's Principle states that two collections have the same number just in case they are equinumerous, in the sense that they can be correlated one-to-one:

(HP) $\#xx = \#yy$ iff $xx \approx yy$.

The principal aim of this article is to use the notion of grounding to develop this sort of abstractionism. The appeal to grounding enables a unified response to the two main challenges that confront abstractionism. First, we must explicate the metaphor of metaphysical "cheapness." Second, we must rebut the "bad company" objection, which rejects abstraction principles like (HP) as tarnished by their similarity to inconsistent principles like Frege's Basic Law V. By enforcing a simple requirement that all abstraction be properly grounded, we propose a unified solution to these two hard, and prima facie unrelated, problems. On our view, grounded abstraction simultaneously ensures "cheap" abstracta and permissible abstraction.

1 Introduction

Many philosophers and other thinkers have been attracted to the idea that some objects are metaphysically "cheap", as their existence does not demand very much of reality. An influential version of this idea appeals to *abstractionist* views in the philosophy of mathematics, on which numbers and other mathematical objects are somehow abstracted from other phenomena.[1]

Contemporary discussions of abstractionism often focus on what's called *Hume's Principle*, which states that two collections have the same number just in case they are *equinumerous*, in the sense that they can be correlated one-to-one. For reasons that will become clear, we prefer a plural version of this principle:

(HP) $$\#xx = \#yy \;\leftrightarrow\; xx \approx yy$$

[1] Aristotle (*Metaphysics* M, Ch.s 2,3 and *Physics* II.2), [Wright, 1983], and [Hale and Wright, 2001]. A different, more radical version of the idea that mathematical objects are "cheap" is associated with the great mathematician David Hilbert, who claims that the mere consistency of a mathematical theory suffices for there to exist objects described by the theory. See his letter to Frege of 29 December 1899, in [Frege, 1980], as well as [Ewald, 1996, p. 1105].

where xx and yy are plural variables, $\#xx$ is a term for the (cardinal) number of individuals in xx, and $\approx$ relates xx and yy just in case there is a one-one function from xx onto yy.

The idea motivating this sort of abstractionism is that true instances of the left-hand side of (HP), which entail the existence of numbers, are in some sense *founded on* the existence of the one-to-one correspondences reported by true instances of the right-hand side and *do not go substantially beyond* them. The existence and arithmetical features of the objects in question are "thin," in the sense that they make no substantial demands on how things are, beyond those demands already made by the corresponding instances of the right-hand side of (HP).[2] On the view on offer, the existence of 2, for instance, is "thin" in the sense that it demands nothing "substantially beyond" what's demanded by there being two ordinary things, and those things' being, collectively, in one-one correspondence with themselves. Similar ideas have been applied to similar principles governing other mathematical entities, including ordinal numbers and sets.

The principal aim of this article is to use the notion of *grounding* to develop this sort of abstractionism. More specifically, we argue that the appeal to grounding enables a unified response to the two main challenges that confront abstractionism.

The first challenge concerns the series of metaphors that have been employed: "cheap," "undemanding," and "thin". While these metaphors are suggestive and plausible, they require clarification. We will explore the prospects for explaining them by appeal to grounding. Very roughly, the idea is that some objects are undemanding or thin relative to some others just in case the existence and basic features of the former objects are grounded in facts about the latter. (HP) illustrates the proposal. On the view we will develop, the existence and arithmetical features of cardinal numbers are grounded in equinumerosity facts concerning pluralities of objects.[3] In a slogan: *grounded abstraction yields thin abstracta.*

The second challenge is known as the "bad company problem".[4] (HP) belongs to a more general class of so-called *abstraction principles*:

(AP) $$\S\alpha = \S\beta \leftrightarrow \alpha \sim \beta$$

where α and β are variables (singular, plural, or second-order), § is a term-forming operator, and $\sim$ is an equivalence relation. This class has some good instances. For example, (HP) is both plausible and provably consistent.[5] Regrettably, these good instances are surrounded by

[2] See [Schwartzkopff, 2011], [Linnebo, 2018], as well as [Rosen, 2011].

[3] For ease of expression, we sometimes talk about "pluralities" where strictly speaking we should be using the plural idiom.

[4] See, e.g. [Studd, 2016], as well as [Linnebo, 2009] for an overview.

[5] See [Boolos, 1987].

"bad companions," namely instances of (AP) that are inconsistent or unacceptable for subtler reasons.[6] How can we separate the good from the bad? We explore the idea that a large and natural class of permissible forms of abstraction can be defined by carefully enforcing a simple requirement to the effect that, roughly, the existence and basic features of the objects abstracted are properly grounded in facts involving only objects already introduced. In a slogan: *grounded abstraction is permissible abstraction.*

Thus, by enforcing our simple requirement that all abstraction be properly grounded, we aim to provide a *unified solution* to two hard, and prima facie unrelated, problems. Grounded abstraction simultaneously ensures thin abstracta and permissible abstraction.

We are not the first to propose that grounding can shed light on abstractionism.[7]. In a sympathetic but critical discussion of this proposal, [Donaldson, 2017] identifies several substantial problems. Plausible principles governing grounding appear to come into conflict with the commitments of the sort of ground-theoretic abstractionism on offer. We propose solutions that draw heavily on the notion of *weak ground* [Correia, 2010],[Fine, 2012a], [Fine, 2012b]. So we also take the opportunity to attempt to explain that relatively unfamiliar idea. Thus, our discussion, if successful, commends the notions of ground and, in particular, of weak ground to abstractionism's proponents.

Here's the plan. In §§2-3 we motivate ground-centered abstractionism. §§4-6 outline three problems flagged by Donaldson. We appeal to weak ground to solve the thorniest of the problems, and so §§7 explains the idea, while §§8-9 apply it to yield a solution. §10 extends the resulting form of abstractionism in order to solve the bad company problem, before §11 concludes.

2 Why an asymmetric conception of abstraction?

Ever since Frege, it has been customary to take a highly symmetric view of abstraction principles, at least as far as their content is concerned. Frege writes that one side of such a principle "carves up the content in a way that is different from the [...] way" the other side does [Frege, 1953, §64]. This is very suggestive. An instance of the right-hand side expresses a certain content, which is "carved up" in a certain way. But that content need not be "carved up" in this particular way. The very same content admits of different "recarvings", such as the one effected by the left-hand side. Similar ideas figure centrally in the work of the neo-Fregeans Bob Hale and Crispin Wright, who have tried hard to articulate the relevant notion of recarving

[6] E.g., there are pairs of principles that are individually consistent but jointly inconsistent; see the works cited in fn. 4.

[7] See [Rosen, 2011], [Schwartzkopff, 2011], and [Donaldson, 2017]

of content.[8]

More recently, Agustín Rayo (2013) has made a fresh attempt to articulate and defend a symmetric conception of abstraction. He begins with the idea that every meaningful statement makes some demands on the world: there are certain ways the world must be for the statement to be true. Consider two knives and two forks. The statement that the number of the former is identical with the number of the latter makes the very same demands on the world as those made by the claim that there is a one-one correspondence between the knives and the forks. Let $\Leftrightarrow$ represent sameness of demands on the world. Then Rayo endorses the universal closure of:

$$\#xx = \#yy \;\Leftrightarrow\; xx \approx yy$$

This epitomizes the symmetric conception of abstraction.

In the introduction, we invoked a contrasting, *asymmetric* conception of abstraction. True instances of the left-hand side of an abstraction principle are in some sense "founded on" the corresponding instances of the right-hand side; for this reason, the former do not go "substantially beyond" the latter. On this asymmetric picture, we do not claim that the two sides of an abstraction principle make identical demands on the world, but that the demands made on the right-hand side *suffice*, in some sense, for those made on the left-hand side. Let us write '$\Rightarrow$' for this so far entirely schematic notion of sufficiency. Allowing this arrow to be turned around, we thus want the universal closure of:

$$\#xx = \#yy \;\Leftarrow\; xx \approx yy$$

Why go against the tradition deriving from Frege and pursue an asymmetric conception? It must be admitted that the symmetric conception is attractively simple. Surely, the simplest way for one demand not to go substantially beyond another is for it not to go beyond at all. This simplicity comes at a high cost, however—or so it seems.[9] For one thing, the left-hand side of (HP) does seem to demand more of the world than the right-hand side. In particular, the left-hand side seems to demand that there be numbers. The two sides have different ontological commitments, at least in the ordinary Quinean sense of that phrase, since the left-hand side, unlike the right-hand side, entails that something is (identical to) the number of xx. For another, we suggest that the asymmetric conception enables a distinctive response to the bad company problem. That response is to insist that the ontology in question be accounted for in a strictly bottom-up manner, where instances of the right-hand side are successively used to

[8] See [Hale and Wright, 2001], especially Ch. 4, as well as [Fine, 2002, pp. 39–41] for criticism.

[9] [Linnebo, 2018, ch. 4] develops the two considerations that follow in more detail.

ground instances of the left-hand side. While we do not claim that the symmetric conception is doomed, its apparent drawbacks provide ample reason to explore the alternative, asymmetric conception.[10]

3 A path to grounding finite cardinal arithmetic

Let us explore, therefore, the idea that the existence and arithmetical features of cardinal numbers are "thin" in the sense that there is a bottom-up ground for the abstract ontology involving the existence and features of pluralities of more familiar, concrete individuals.

The most straightforward version of this idea would be to take sufficiency to be a matter of ground. Thus, in particular, we would have:

$$xx \approx yy \ < \ \#xx = \#yy$$

Here, as is standard in the literature, $<$ is a sentential operator expressing *full, strict* grounding. A full ground for a given fact f is a collection of facts that, together, need no supplementation to ground f. If we list those facts, we have offered a complete answer to the question of what it is in virtue of which f obtains, with no supplementation required. Thus, if asked what makes it the case that a certain figure is a square, a complete answer might appeal to a pair of facts: that it is equilateral, and that it is rectangular. By contrast, if we just said that it is rectangular, we have given at most a partial answer, requiring supplementation. *Strict* ground is the notion typically used and discussed in the literature: a strict ground for f is a collection of facts that make f the case. We develop the contrast between strict ground and the counterpart idea of a *weak* ground below.[11] Syntactically, the sentential operator $<$ takes many sentences on the left and a single sentence on the right. A sentence of the form $\phi_0, \phi_1, \dots < \psi$ says that what makes it the case that ψ are (collectively) $\phi_0, \phi_1, \dots$.

There is an immediate problem with the proposed identification of sufficiency with strict

[10]While we use grounding to develop the asymmetric conception, another option is to use the resources of modal logic; see [Linnebo, 2018], [Studd, 2016], and [Studd, 2019]. Instead of saying that the existence of a cardinal number is *grounded* in that of an appropriately numerous plurality, the modal approach says that the existence of the number is *merely potential*, in some sense, relative to that of the plurality. Each approach has philosophical commitments that go beyond those of the other. Most importantly, where our present approach is committed to the intelligibility of the very general notion of grounding, the modal approach is based on a more specific account of permissible expansions of one's language. ($\Diamond\varphi$ represents that it is permissible to expand the language so as to make φ true; $\Box\varphi$ makes the dual claim.) We believe, however, that the two approaches can be rendered consistent. For one, [Linnebo, 2018, p. 18] proposes that his explication of sufficiency, which his modal operators are introduced to represent, can be seen as "a species of metaphysical grounding"; see also pp. 43–4 n. 41, 56 n. 18, and 193. For another, as Rayo (p.c.) suggests, it might be possible—even if not our preferred choice here—to interpret the *formal* machinery of grounding in terms of principles governing the dynamics of permissible language expansions.

[11]See [Fine, 2012a] for discussion of these two distinctions.

full ground. Intuitively, grounding is factive: if the fact that the number of forks is identical to the number of knives is grounded in (that is, obtains in virtue of) the fact that there is a one-one correspondence between forks and knives, then it follows that there is such a correspondence, and that the numbers of the two groups of utensils are identical. However, sufficiency of the relevant sort is intuitively not factive. The existence of a law against eating ice cream is presumably sufficient, in the relevant sense, for ice cream consumption to be a crime. Fortunately, there is no such law, so we need not fear criminal sanctions for indulging in Chunky Monkey. Thus, sufficiency cannot just be identified with grounding. But it can be characterized in this case as *conditional* grounding.[12] That is, in the case of (HP), we might express the sufficiency of equinumerosity for arithmetical identity thus:[13]

$$(\#<) \qquad xx \approx yy \rightarrow (xx \approx yy < \#xx = \#yy)$$

A similar idea expresses sufficiency in the negative case:

$$(\neg\#<) \qquad xx \not\approx yy \rightarrow (xx \not\approx yy < \#xx \neq \#yy)$$

Armed with these claims, we appear to get bottom-up grounds for the ontology of arithmetic. Let's, for the time being, take on board two widely presupposed ideas. First, let's suppose that $\phi(a)$, if true, is a full ground for $(\exists x)\phi(x)$. Second, let's suppose that grounds chain, so that any ground for a ground for ϕ is a ground for ϕ.[14] Let aa be Biden. Then we have:

$$aa \approx aa < \#aa = \#aa < (\exists x)x = \#aa.$$

We can then let bb be the pair of Biden and $\#aa$, and get:

$$bb \approx bb < \#bb = \#bb < (\exists x)x = \#bb$$

And so it goes. Thus, $(\#<)$ and $(\neg\#<)$ appear to give us grounds for an infinite ontology of cardinal numbers.

Following much of the literature on abstraction, we employ a negative free logic, where identity statements are taken to be existence involving.[15] Thus, '$t = t$' can be understood as

[12] Alternatively, we might characterize sufficiency by appeal to a primitive notion of *non-factive* grounding, of the sort discussed at [Fine, 2012b, pp. 48-50].

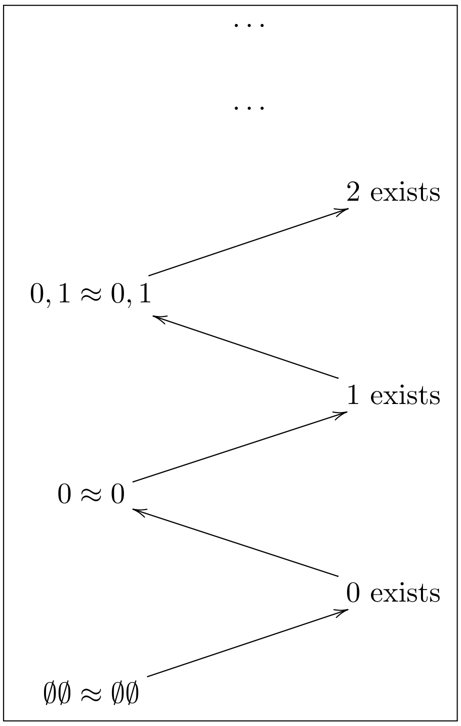
[13] Essentially the same principle is known in some of the literature as the "Schwartzkopff-Rosen principle".

[14] Technically, what we are presupposing is a CUT principle to the effect that, if $\phi_0, \phi_1, \ldots, \Gamma < \psi$ and, for each i, $\Delta_i < \phi_i$, then $\Delta_0, \Delta_1, \ldots, \Gamma < \psi$; see [deRosset, 2014], adapting [Fine, 2012a]. Here comma-delimited lists are used to indicate unions of pluralities. Thus, we write ϕ, Δ for the plurality containing just ϕ and the members of Δ.

[15] See e.g. [Hale and Wright, 2009, pp. 463–64] and [Linnebo, 2018, Appendix 2.B].

asserting the existence of t. So we will sometimes informally say "t exists" in place of '$t = t$'. In addition to being fairly standard, the choice of a negative free logic has the advantage of allowing a pleasingly simple and direct integration of abstraction and grounding.

Things are even better if we accept an empty plurality $\emptyset\emptyset$ of no objects whatsoever.[16] Then it appears that we can bootstrap the entire ontology of finite cardinal numbers from the presumably trivial and necessary fact that $\emptyset\emptyset \approx \emptyset\emptyset$. In particular, we have grounds for the existence of an initial element 0, defined as $\#\emptyset\emptyset$. Now, the singleton plurality of 0 is obviously equinumerous with itself. What grounds this equinumerosity fact? It is natural to think that the mere existence of 0 provides a full account of why the corresponding singleton is self-equinumerous. We will examine this thought shortly, but for now, simply suppose that it is true. Then we can continue in this way in familiar Fregean fashion. Suppose we have obtained grounds for the existence of the numbers $0, 1, \ldots, n$. Let nn be the plurality of all these numbers. Again, it is natural to think that the mere existence of this plurality grounds its self-equinumerosity. Then the self-equinumerosity of nn provides a ground for the existence of $n + 1$. In short, it appears we can bootstrap grounds for the entire ontology:



We also adopt clauses for sufficiency of the sole non-arithmetical predicate of this Fregean form of arithmetic, namely '$P(x, y)$', for "x immediately precedes y". Let '$\textsc{Prec}(xx, yy)$' state that the result of removing one of yy is equinumerous with xx. Then the principles in question

[16] Although ordinary English does not recognize such a plurality, arguably there are legitimate languages that do. See [Florio and Linnebo, 2021, p. 63] for discussion. Another option would be to let the work currently done by plurals be done by second-order logic restricted to properties that are "extensionally definite"; see fn. 53.

are:[17]

$$(P{<}) \qquad \textsc{Prec}(xx, yy) \rightarrow (\textsc{Prec}(xx, yy) < P(\#xx, \#yy))$$

$$(\neg P{<}) \qquad \neg\textsc{Prec}(xx, yy) \rightarrow (\neg\textsc{Prec}(xx, yy) < \neg P(\#xx, \#yy))$$

In this way, we can ground facts such as $P(2, 3)$ and $\neg P(3, 2)$.

On the grounding-based abstractionist view we are discussing, cardinal numbers are "thin" in the sense that their existence and atomic properties (or negations thereof) are grounded, without remainder, in the existence and properties of more familiar things like Biden. To ground this entire infinite system, the most God needs to do is to create Biden or some other individual. In this sense, the existence of Biden "suffices" for the existence of the entire infinite system of cardinal numbers, and the existence of that system does not impose any substantial demands on the world other than those imposed by the claim that something—anything—exists.

Even better, if we accept the empty plurality $\emptyset\emptyset$, then no action is needed by God to ensure that each of the finite cardinal numbers exists, so long as They need take no action to ensure that $\emptyset\emptyset \approx \emptyset\emptyset$. Some philosophers would regard this as a form of "zero-grounding" – that is, a form of grounding that requires no input whatsoever.[18] However it may be understood, this form of grounding allows a nice explication of the idea of objects that are absolutely "thin": these are objects that are grounded with no input whatsoever. Other abstract objects, by contrast, would be relatively, but not absolutely, "thin." As an example, consider Frege's famous claim that directions are abstracted from lines, with two lines giving rise to the same direction just in case the lines are parallel [Frege, 1953, §64]. The existence of a direction would then be grounded in the self-parallelism of any suitable line. Thus, to ensure that directions exist, God needs to ensure that there is space in the first place, which would then contain lines suitable for abstracting the direction in question.

Returning to our main example, we seem to have a path to a grounding of all facts expressed by atomic or negated atomic truths of first-order, finite cardinal arithmetic. This is *very* promising. There are, however, some complications. We will now discuss, in order of increasing seriousness, three complications identified in [Donaldson, 2017].

[17] Suppose we define $P(x, y)$ as $\exists xx \exists yy(\textsc{Prec}(xx, yy) \wedge x = \#xx \wedge y = \#yy)$. Then the first, but not the second, principle can be derived, using the logic of ground.

[18] See [Fine, 2012b] and [Litland, 2017].

4 The individuation of arithmetical facts

The first complication concerns the individuation of arithmetical facts. Suppose aa are just Biden, and bb, just Harris. The ground-theoretic gloss we have given (HP) yields a view about what grounds such facts as

(1) $\#aa = \#bb$.

But what about the following, closely related arithmetical facts?

(2) $1 = \#aa$.

(3) $1 = 1$

Here we face a choice concerning the relevant arithmetical facts. Donaldson distinguishes two options. One option is that the three mentioned facts are identical and thus have the very same grounds; as does any instance of

(4) $\#\tau\tau = \#\tau\tau$

where $\tau\tau$ is a plural term referring to a singleton plurality. It follows that ($\#<$) yields grounds for $1 = 1$. For if aa are just one, then by ($\#<$) its equinumerosity with itself grounds $\#aa = \#aa$, which is the very same fact as $1 = 1$. Obviously, we should extend the same courtesy to all of the other positive cardinal numbers (and to 0 if we allow $\emptyset\emptyset$).

Another option is that (1), (2), and (3) do not express the same fact. Then we don't yet have a view of how identities of pure arithmetic, such as (3), are grounded. Nor do we have a view of how arithmetical existence facts such as

(5) $(\exists x)x = 1$

are grounded.[19]

The former view is appealingly simple. By taking a range of different sentences to express the same fact, it ensures that the simple account of arithmetical grounding laid out in the previous section suffices. However, the latter view *seems* more plausible. After all, (1) plausibly involves Harris somehow, whereas (2) and (3) do not.

Which view is correct? The answer depends, at least in part, on the appropriate analysis of abstraction terms such as '$\#xx$'. Suppose these terms contribute to the facts expressed in a "purely referential" manner in the sense of [Quine, 1960, 177]. Suppose, that is, that these

[19]Donaldson [2017] refers to these options as the *coarse* and the *fine* views, respectively. We prefer to avoid this terminology, to forestall any conflation of the current question with the orthogonal question of how finely individuated facts in general are.

terms are used merely to pick out certain objects, which are then plugged into the relevant facts. For example, if we augment English with the function term '$\textsc{husband-of}$', then on the purely referential analysis

(6) $\textsc{husband-of}(\text{Michelle}) = \text{Barack}$

expresses the same fact as

(7) $\text{Barack} = \text{Barack}$.

For the complex term '$\textsc{husband-of}(\text{Michelle})$' serves merely to pick out Barack, whom the rest of (6) asserts to be identical with Barack. Likewise, on this analysis, all of the arithmetical sentences (1)–(4) express the same fact. Alternatively, if functional complex terms are not given a purely referential analysis, then '$\textsc{husband-of}(\text{Michelle})$' does not serve simply to plug its referent into the fact expressed by (6), and so that fact, unlike the fact expressed by (7), plausibly involves Michelle. On that analysis, then, (6) and (7) are most naturally taken to express distinct facts.

Our question, then, is whether terms of the form $\#tt$ should be given a purely referential analysis. As an interpretation of semi-ordinary English, we admit that the non-purely referential analysis has considerable plausibility. To keep things simple, however, we propose to stipulate that the relevant complex terms be given a purely referential analysis—with the result that all of the arithmetical facts (1)–(4) with which we opened are identical. We are free to make this stipulation because terms of the form '$\#tt$' do not occur in ordinary English. Of course, English contains definite descriptions like 'the number of planets', but that is a separate matter. Since the functional terms '$\#tt$' are introduced by us, we are free to stipulate how they are to be analyzed. And it is expedient for us to stipulate that they be given a purely referential analysis.

Before leaving the question of the individuation of arithmetical facts, we wish to make three remarks. The first concerns applied arithmetic, where we will sometimes need to express the fact that some objects xx are n in number. The obvious proposal is to express this as '$\#xx = n$'. While this works well on the non-purely referential analysis, it misfires on the purely referential one, where '$\#xx = n$' and '$n = n$' express the same fact. We therefore propose to add a predicate '$\textsc{Num}(xx, n)$', subject to the principle that $xx \approx xx$ grounds $\textsc{Num}(xx, \#xx)$—or, equivalently, using pure referentiality, $\textsc{Num}(xx, n)$ for $n = \#xx$—and *mutatis mutandis* for the negated case.

Second, the question of whether to adopt the purely referential analysis of functional terms could, if desired, be sidestepped by rewriting the principles concerning the grounding of arithmetical facts in a way that replaces functional terms with variables, whose semantic contribution is certainly purely referential.[20] For example, ($\#<$) can be rewritten as the universal closure of:

$$(\#< \text{exp}) \qquad x = \#uu \wedge y = \#vv \wedge uu \approx vv \;\rightarrow\; (uu \approx vv < x = y)$$

and likewise for our other claims about the grounding of arithmetical facts.

Third, suppose we made the reverse stipulation, namely, that the function terms be given the non-purely referential analysis. It might still be an option to allow the resulting distinct facts expressed by (1)–(4) to share their grounds. If so, the grounding-based abstractionist could affirm in good conscience that these facts are grounded in the same way.

5 The grounding of equinumerosity facts

The second and third problems identified by Donaldson are manifestations of *the impredicativity* of Hume's Principle, by which we mean that the principle quantifies over all objects whatsoever, including the numbers whose existence it is supposed to explain. Let us explain what we have in mind, starting with the second problem.

This problem turns on a question that we have so far glossed over: how are true equinumerosity statements, or true negations thereof, grounded? It is standard to define equinumerosity in terms of unrestricted quantification: $aa \approx bb$ iff there is a relation R that is functional, total on aa, one-to-one, and onto bb. Each of the requisite features of R is, in turn, defined by unrestricted quantification. For instance, the claim that R is total on aa is analyzed as:[21]

$$(\forall x)(x \in aa \rightarrow (\exists y)(y \in bb \wedge R(x, y))$$

As Donaldson shows, if we assume that generalizations are grounded in their instances, we get chains of grounding relations that appear to go in the "wrong" direction, taking us *up* the natural numbers instead of *down* to more basic, non-arithmetical facts.

To analyze the problem, we will need to appeal to some further principles regarding grounds for facts expressed by logically complex sentences. Let's express the notion of *partial ground* using the operator $\prec$. Intuitively, then, $\phi \prec \psi$ says that the fact expressed by ϕ is a (perhaps improper) part of some collection of facts that, together, fully ground ϕ.[22] Here are the

[20] This transformation is known as 'exportation' [Quine, 1956], [Kaplan, 1968].

[21] In the literature on plural logic, '$\prec$' is often used for plural membership. Since this symbol is already in use in the grounding literature, we take inspiration from set theory and write '$x \in aa$' instead of '$x \prec aa$'.

[22] This is the notion of *partial strict* ground defined in [Fine, 2012a].

further principles required for the statement of the problem:

$(\vee <)$ $$\phi \to (\phi < \phi \vee \psi) \text{ and } \psi \to (\psi < \phi \vee \psi)$$

$(\wedge <)$ $$(\phi \wedge \psi) \to (\phi, \psi < (\phi \wedge \psi))$$

$(\to\ <)$ $$\Delta < (\phi \to \psi) \leftrightarrow \Delta < (\neg\phi \vee \psi)$$

$(\exists \prec)$ $$\phi(a) \to (\phi(a) \prec (\exists x)\phi(x))$$

$(\forall \prec)$ $$(\forall x)\phi(x) \to (\phi(a) \prec (\forall x)\phi(x))$$

Though there is significant controversy in the literature about the validity of these schemas, they are still widely presumed.[23]

The principle $(\forall \prec)$ says that the features of any entity whatsoever are among the grounds of facts expressed by unrestricted universal quantifications. Since the definition of $\approx$ uses such quantification, this means that larger cardinal numbers get involved in grounding the self-identity and existence of smaller ones. Assume, as is plausible, that if $x \notin aa$ and $a \in aa$, then $x \neq a$ is a partial ground for $x \notin aa$. Let aa consist of Biden and the number 1. Then, on our assumptions, arithmetical facts involving 3 partially ground the fact that $2 = 2$. For we have:

$$
\begin{array}{lll}
 & 2 = 2 & \\
\succ & aa \approx aa & (i.e., (\exists f) f : aa \xrightarrow[onto]{1-1} aa) \\
\succ & =: aa \xrightarrow[onto]{1-1} aa & \\
\succ & (\forall x)(x \in aa \to (\exists y)(y \in aa \wedge y = x)) & (i.e., =: aa \xrightarrow[onto]{} aa) \\
\succ & (3 \in aa \to (\exists y)(y \in aa \wedge y = 3)) & \\
\succ & 3 \notin aa & \\
\succ & 3 \neq 1. &
\end{array}
$$

This seems problematic because the grounding chain is, intuitively, proceeding upwards, rather than downwards, as one would expect. More generally, it is at least as discomfiting that the features of any entity whatsoever will figure among the grounds of $2 = 2$ on our assumptions. These problems stem from the use of unrestricted quantification in the definition of '$\approx$'. To get a view more in the spirit of the abstractionist viewpoint we are exploring, something more constrained is required.

This brings us, finally, to our choice to work with the plural version of (HP), rather than the usual property-based version. For a plural equinumerosity statement $aa \approx bb$ seems to be, intuitively speaking, *intrinsic* to aa and bb. (The idea here is that how things stand with aa and bb suffices to fully ground $aa \approx bb$ or its negation, with no need to consult the rest of

[23] See [Fine, 2012b, Rosen, 2010] for influential endorsements, and [deRosset, 2021], [Fine, 2010], [Litland, 2015], [McSweeney, forthcoming], [Thompson, 2016], and [Hofweber, 2009] for various criticisms.

reality.[24]) Contrast the correlative notion of equinumerosity for *properties*. An equinumerosity statement of the form $F \approx G$ turns on all of reality. For instance 'STARS $\approx$ PLANETS' requires us to consider, for any object whatsoever, whether this object is a star (or a planet). So, every time a new star ignites, 'STARS $\approx$ PLANETS' may change its truth value. The case of pluralities is different.[25] Suppose ss are all and only the stars and pp are all and only the planets. Then $ss \approx pp$ is intrinsic to ss and pp: there is no need to consider any objects other than precisely ss and pp. Regrettably, as Donaldson's argument shows, our present analysis of $ss \approx pp$ does not reflect this contrast.

We propose to solve this problem by defining $\approx$ using a primitive notion of quantification *restricted to a plurality.*[26] For instance, we take $(\forall x \in aa)\varphi(x)$ to deploy primitive, restricted quantification, rather than taking it to be shorthand for the unrestricted quantificational claim $(\forall x)(x \in aa \to \varphi(x))$. The difference in the meanings of the two claims shows up in a difference in how the facts they express are grounded: it is plausible to think that facts expressed by generalizations restricted to a plurality are partially grounded only in $\varphi(a)$ for each $a \in aa$, but not that they are partially grounded in each claim $(a \in aa \to \varphi(a))$ for any a whatsoever. Consider, for instance, the fact that each of Marcus, Kripke, and Williamson is a philosopher. It seems that what grounds this fact is just whatever makes it the case that Marcus is a philosopher, Kripke is a philosopher, and Williamson is a philosopher. Other philosophers (and their distinctness from Marcus, Kripke, and Williamson), much less stars, planets, or the number 3, do not enter into the question. Similar remarks apply to restricted existential generalizations. The fact that at least one of Marcus, Kripke, and Williamson is alive is grounded in whatever makes it the case that either Marcus is alive, Kripke is alive, or Williamson is alive. This proposal renders the argument for $3 \neq 1 \prec 2 = 2$ above unsound, since $3 \notin aa$ is not a partial ground of the restricted universal generalization that, for each x of aa, at least one y of aa is identical to x.

Donaldson proposes appealing to a similar idea to solve the problem, on which the fact expressed by a true restricted universal generalization of the form 'Every F is G' is grounded only in the facts expressed by its instances Ga for each object a that is F [Donaldson, 2017, Sect. 6]. This proposal gives rise to cases in which we have grounds for the fact expressed by a restricted generalization that do not necessitate that fact. According to the proposal, grounds for the fact that every star is no brighter than Sirius include that Sirius is no brighter, that

[24]In Appendix C, this informal idea is replaced by a precise technical notion, capable of bearing the argumentative weight on its own.

[25]There are other reasons too to prefer a plural version of (HP); see e.g. [Schwartzkopff, 2011, pp. 357-62].

[26]We might instead take $\approx$ to be primitive. But there seems to be a perfectly appropriate analysis of it, specified below, in terms of bijective functions. Moreover, our solutions to the problems below depend on that analysis. Thanks to an anonymous referee for this suggestion.

Betelguese is, that Alpha Centauri is, *etc.* If a new, brighter star ignites, then the grounds still obtain but the generalization is no longer true. The proposal is, for this reason, highly controversial.[27] It also fails to provide what we need in our technical argument for Proposition 1 below. Notice that, by contrast, the proposal concerning generalizations restricted to pluralities does not require this counter-example to the claim that grounds necessitate what they ground. For instance, the grounds for the fact that each of ss is no brighter than Sirius are, on our proposal, that Sirius is, that Betelguese is, *etc.* In the situation in which a new star ignites, the generalization is still true, since the new star is not among ss.

We therefore suggest that, instead of $(\forall \prec)$ and $(\exists \prec)$, we use:

$(\exists_{\boldsymbol{aa}} <)$ For any $a \in aa$, $\phi(a) \to (\phi(a) < (\exists x \in aa)\phi(x))$; and

$(\forall_{\boldsymbol{aa}} <)$ For any $aa = a_1, a_2, \ldots$, $((\forall x \in aa)\phi(x) \to (\phi(a_1), \phi(a_2), \cdots < (\forall x \in aa)\phi(x))$.

For similar reasons, we propose analogous principles for quantificational claims of the form $(\exists xx \subseteq aa)\phi$ and $(\forall xx \subseteq aa)\phi$; see Appendix A.

We are now in a position to solve Donaldson's second problem. An equinumerosity statement $aa \approx bb$ can be analyzed using only quantification restricted to the pluralities aa and bb, as well as higher-order analogues of such quantification. (The analysis is spelled out in Appendix A.) Analyzed in this way, the equinumerosity statement involves only the objects aa, bb. To ground its truth or falsity, no other objects need to be invoked. Indeed, we can prove:

Proposition 1 Given the logical grounding principles stated above, $aa \approx bb$, if true, is fully grounded in all the truths of the form $c = c'$ and $c \neq c'$ where $c, c' \in aa$ or $c, c' \in bb$; and likewise for $aa \not\approx bb$.

(The proof, which contains no conceptual surprises and is largely a matter of bookkeeping, is spelled out in a supplement to this article.)

6 The problem of autoabstraction

We have made considerable progress, but a final problem identified by Donaldson remains. Even granting our assumptions concerning the grounding of equinumerosity claims, any view that affirms $(\# <)$ faces what has become known as the *problem of autoabstraction* [Zanetti, 2020]. In essence, the problem is simple. Let cc be the plurality whose single member is the cardinal number 1. The equinumerosity of this plurality with itself is partially grounded in the

[27] See [deRosset, 2010], [deRosset, forthcoming, §8.4.2], [Leuenberger, 2014], [Skiles, 2015], and [Trogdon, 2013] for discussion.

existence of 1. Yet by ($\#<$), this equinumerosity is also a ground of the existence of 1. So we get an unacceptable cycle of ground.

For a proper analysis of the problem, we need to make two assumptions explicit. The first is that ground is *acyclic*, in the strong sense that nothing is even a partial ground of itself. This assumption is highly plausible.[28] A claim of the form

(8) ϕ in virtue of the fact that $\phi, ...$

seems clearly false. Thus, any ethicist who proposed that a certain act a is wrong partly in virtue of being wrong would face an immediate charge of implausibility. As with the other principles, there is controversy in the literature about whether ground is acyclic, but it is also widely presumed.[29] Second, the problem rests on the assumption that partial ground is transitive: partial grounds of partial grounds are themselves partial grounds. Again, this is widely presumed, though there are skeptics.[30]

We can now present, in proper detail, the problem outlined above. Let cc have 1 as its single member. Then $\#cc \in cc$: that is, 1 is, intuitively, abstracted from itself. Using **id** for identity restricted to cc, our grounding principles yield:

$$\begin{array}{lc} & 1 = 1 \\ \succ & cc \approx cc \\ \succ & \mathbf{id} : cc \underset{onto}{\xrightarrow{1-1}} cc \\ \succ & (\forall x, y, z \in cc)(x = y \wedge z = y \rightarrow x = z) \\ \succ & (1 = 1 \wedge 1 = 1 \rightarrow 1 = 1) \\ \succ & 1 = 1. \end{array}$$

[Donaldson, 2017]. Thus, we obtain the advertised cycle of ground.

This problem is another manifestation of the impredicative character of Hume's Principle, noted in the previous section. There we proposed an analysis of equinumerosity statements $xx \approx yy$ as intrinsic to xx and yy. Although that is progress, we are not yet done. For these pluralities may themselves have as members the number whose existence we are trying to explain. As a result, interpreting Hume's Principle in ground-theoretic terms leads to cycles of ground: 2 exists partly in virtue of the existence of 1 and vice versa. So, on pain of commitment

[28] A comparatively moderate departure from this assumption would accept cyclic grounding in cases where there is also an acyclic (or perhaps even well-founded) way to ground the relevant truth. After all, 1 depends only generically on singleton pluralities: any such plurality will do. Thus, although we may reach 1 by passing through that very object, as in the above example, there are also noncircular ways to reach 1, e.g., by abstracting on the singleton plurality of Biden [Zanetti, 2020]. In our opinion, it would be preferable not to be forced to make even this moderate departure.

[29] See [Correia, 2014], [Krämer, 2013], and [Woods, 2018] for criticism.

[30] See [Schaffer, 2012] and [Tahko, 2013] for arguments against transitivity, and [Litland, 2013] for a defense.

to cycles of ground, the proposal to use (HP) as a guide to giving a bottom-up account of how the ontology of arithmetic is grounded, and thus "thin," appears hopeless.

What to do? One quick solution immediately suggests itself: we have suggested ($\#<$) and ($\neg\#<$) as a means of grounding the entire ontology of (finite) cardinal numbers, but one might hope that something a little weaker will do. An obvious alternative is to suggest that our account of the grounds for the key arithmetical facts follow a standard, bottom-up approach where 1 is defined as the cardinal of the singleton plurality of 0, then 2 as the cardinal of the pair $0, 1$, and so on.[31]

This response avoids a cycle of ground, but it creates another problem. On general abstractionist principles, there are other ways to get cardinal numbers off the ground. For instance, we could first abstract 2 and then work our way down and up from there. To illustrate, an *unconventional abstractionist* might first define **2** as the number of the pair of Biden and Harris, next define **1** as the number of the singleton of **2**, then **3** as the number of the triple of Biden, **1**, and **2**, and so on. This step-by-step constructive procedure for grounding arithmetical identities may be inelegant, but it is in perfectly good order.[32] To avoid equivocation, we will put the unconventional abstractionist's numerals in boldface. We now face an awkward question: is $2 = \mathbf{2}$?

It would seem not. On the view under discussion, something grounds the fact expressed by '$2 = 2$' that does not ground that expressed by '$\mathbf{2} = \mathbf{2}$'. This difference requires that we distinguish the facts in question, which, in turn, requires that we distinguish 2 from **2**. Similar arguments yield similar results for the other finite cardinals. Obviously, the unconventional abstractionist has a counterpart who starts with a triple rather than a pair, another who starts with a quadruple rather than a pair, and so on. So, we seem to be able to abstract any number of "copies" of the cardinals by starting with different numbers and working down and then up, as illustrated above. The proposal thus engenders an unlovely and implausible fragmentation of the ontology of arithmetic.

We now turn to an alternative proposal for an abstractionist account of how the ontology of arithmetic is grounded. On this proposal, we must distinguish between *strict grounds* and *weak grounds* for $1 = 1$. Once we do so, we will see that the connections of ground corresponding to instances of (HP) are not in every case familiar connections of strict ground. In particular, the cycles involved in the problem of autoabstraction are *merely weak*, and so do not implausibly

[31] As Donaldson notes, this response handles only the cases of the finite cardinals. A slight weakening of ($\#<$) might handle infinite cardinals, too:

(✢) If $\alpha = \#xx$ and $\neg(\exists\beta \in xx)(\exists yy)(\beta = \#yy$ and $(\exists f) f : xx \xrightarrow{1-1} yy)$, then $xx \approx xx < \alpha = \alpha$.

Note here that the condition that xx contain neither α nor any larger cardinal is not among the grounds of $\alpha = \alpha$. It is just a condition on the equinumerosity fact's being a ground.

[32] This adapts a point made by Tony Martin (p.c.).

entail that there are cycles of *strict* ground. This threatens to deepen the mystery, rather than solve it. The notion of weak ground is difficult and unfamiliar. So, our proposal faces two urgent questions: (i) what is weak ground? and (ii) how can one motivate the idea that the grounding relations here are merely weak? The answer we propose to the first question provides an answer to the second question.

7 Weak grounding and explanatory levels

The notion of weak ground comes up very naturally for the logic of ground. It was first treated in [Correia, 2010] and [Fine, 2012b]). It is invoked to solve technical problems in the logic of ground [Correia, 2010], [deRosset and Fine, 2023], [Fine, 2012b]. Solving these problems requires that weak ground have a number of features, including that it be reflexive and transitive, and that any strict ground of ϕ also be a weak ground of ϕ. The standard notation for indicating weak ground, '$\leq$', emphasizes these two features.

Unfortunately, requiring that weak ground have these features doesn't nail down the idea. Fine [Fine, 2012a] gives a number of different characterizations. Each of them has problems.[33]

The most intuitively appealing characterization is this: a weak ground for ϕ is something which is *either on the same explanatory level as ϕ or lower.* There seems to be something to this idea. ϕ is, of course, on the same explanatory level as itself, and it is on a lower level than $\phi \vee \psi$. So far, so good. But the notion of an explanatory level is not yet clear enough. $(\phi \vee \psi)$ and $(\phi \vee \chi)$ are, in at least one intuitive sense, on the same explanatory level: each is just above ϕ. But they aren't in general weak grounds for one another.

In other work [Correia, 2017] and [deRosset, 2014] have each proposed that we interpret the idea in a way strongly suggested by the notation $\leq$: A weak ground for ϕ is either (i) a strict ground for ϕ; (ii) identical to ϕ; or (iii) has the form Δ, ϕ, where $\Delta < \phi$. This definition arises from the idea that all that weak grounding essentially adds to strict ground is the fact that each ϕ is to weakly ground itself; add this fact to the strict grounding facts, close under chaining, and we get the weak notion.[34]

This notion plays the theoretical role carved out for weak ground, but it is stronger than

[33]See [deRosset, 2014] for some preliminary reasons for dissatisfaction.

[34]Correia [2010] and Fine [2017] each suggest a view on which weak ground is a matter of propositional identity: $\psi_1, \psi_2, \ldots$ are a weak ground for ϕ iff $(\psi_1 \wedge \psi_2 \wedge \ldots)$ is a *disjunctive part* of ϕ iff $\phi = (\phi \vee (\psi_1 \wedge \psi_2 \wedge \ldots))$. Showing that the resulting notion is reflexive, transitive, and is implied by strict ground requires undertaking a bevy of commitments concerning propositional identities and grounds, detailed by Correia and Fine. We don't intend to take up the case for those commitments here. What's more, it is not clear how this conception provides a sense in which a weak ground for ϕ is "at the same explanatory level as ϕ or below." So, we leave the development and defense of this conception of weak ground and its application to the case of abstraction to others. Thanks to an anonymous referee for indicating the need to note this alternative conception of weak ground.

what is required for the logic of ground most widely presumed in the literature,[35] and Fine (p.c.) objects that it is stronger than the notion he has in mind. For, he argues, if x, y, and z are distinct bodies with the same mass, x's having the same mass as y and y's the same mass as z weakly grounds x's having the same mass as z: they are, intuitively, all on the same level. But neither x's having the same mass as y nor y's having the same mass as z is identical to x's having the same mass as z, nor do they strictly ground x's having the same mass as z.

Again, the notion of levels makes an appearance. If we are to vindicate this idea, we had better get a handle on what that notion comes to. We can do so if we attend to the correspondence between true grounding claims and explanatory arguments, and consider some plausible cases of explanatory arguments.

A highly plausible view about grounding claims $\Delta < \phi$ is that they are true if and only if there is an argument containing only truths whose conclusion is ϕ, whose premises are Δ, and in which the series of inferences from Δ to ϕ follows the order of dependence and determination among the facts reported by those sentences. We may summarize this claim by saying that every grounding explanation is *accompanied* by an argument of the relevant sort.

Why think the claim is true?[36] One particularly nice, particularly convincing way to answer the question, "In virtue of what is it the case that ϕ?" is to cite some truths, and then trace an argument from those truths to ϕ, where the steps in the argument trace the order of dependence and determination. So, for instance, consider

(9) (it's either sunny or snowy) in virtue of the fact that it's snowy.

This claim is true. It is accompanied by the explanatory argument

$$\boxed{\dfrac{\text{It's snowy.}}{\text{So, either it's sunny or it's snowy.}}}$$

Similarly, one way to tell that

(10) It's snowy in virtue of the fact that it's snowy and LeBron James is tall

is not true is to note that

$$\boxed{\dfrac{\text{It's snowy and LeBron James is tall.}}{\text{So, it's snowy}}}$$

while valid, is not an explanatory argument. Its sole inference does not trace any relations of dependence and determination. In fact, in this case the direction of dependence and determination is exactly opposite the direction of inference.[37]

[35]That logic, GG, is axiomatized and semantically characterized in [deRosset and Fine, 2023], following [Fine, 2012b].

[36]Here we follow the motivation in [deRosset, 2021] and [deRosset, forthcoming, §1.2].

[37]Here, following [Correia, 2014] and [Fine, 2010] we have principally in mind explanatory arguments whose inferences involve no discharge of dependencies on premises. In app. A we explicitly claim that certain inference rules have explanatory instances. Those rules are very similar to those used in [Correia, 2014]. None of them

When we offer answers to questions that call for grounding explanations, we construct and criticize arguments of this sort; we train our students to explain things by constructing such arguments, and we identify gaps in their accounts by finding gaps in the corresponding arguments. The activity of asserting and defending explanations involves the production of putatively explanatory arguments. This makes the assumption that true grounding explanations are accompanied by such arguments highly plausible, since that assumption is manifest in our practices of constructing, asserting, and defending explanations.

Say that an argument is *explanatory* when each of its inferences trace the order of dependence and determination. It would be good to have a helpful specification of the conditions under which inferences trace the order of dependence and determination. Unfortunately, there doesn't seem to be any such thing. However, we do have some sense of the idea, which is manifested in our recognition that the first of the two arguments above is explanatory, and the second is not.

Presented with the universe of explanatory arguments for ϕ containing only truths gives us a powerfully intuitive grip on an idea of being at ϕ's explanatory level or below. Let's stipulate that the limit case in which an argument consists simply of a single premise with no inferences is an explanatory argument. (All of its inferences are explanatory.) Further, we will assume that explanatory arguments chain, so that, if there is an explanatory argument from Δ to ψ, and an explanatory argument from ψ to ϕ, then there is an explanatory argument from Δ to ϕ.[38] Then one truth is at or below the explanatory level of another iff the former appears in an explanatory argument for the latter containing only truths. Since being a weak ground for something is being at or below its explanatory level, we have a powerfully intuitive conception of weak ground:

Weak ground: $\Delta \leq \phi$ iff there is a corresponding explanatory argument containing only truths.[39]

allow for discharge of dependencies. We are here neutral on the question of whether some discharge rules have explanatory instances, though admitting explanatory arguments involving discharge would require adjustments in the account of weak ground below. Also, unlike Fine and Correia, we take the existence of explanatory arguments to suffice for *weak* grounding claims, rather than strict grounding claims. As a result, unlike Correia, we do not accept reflexive instances of partial strict ground.

[38] Technically, the assumption is that explanatory arguments obey a CUT principle: if $\Delta_i \vdash \psi_i$ and $(\psi_i) \vdash \phi$ are explanatory, for each $i \leq \alpha$, then $(\Delta_i) \vdash \phi$ is also explanatory. Obviously, many such arguments are not humanly apprehensible. For instance, no human being has the attention or lifespan to appreciate even a one-inference explanatory argument from the heights of the plurality of human beings to the truth about the average height of the members of that plurality, even though that argument is presumably explanatory. Similarly, if there is an explanatory argument from purely physical facts to moral or psychological facts, it is complex enough to exceed our powers of apprehension.

[39] See Appendix A for a more rigorous definition of the notion of an explanatory argument, and of the correlative notion of an explanatory argument's *corresponding* to a weak grounding claim. Note that, since explanatory arguments may contain falsehoods, they are playing a role that is sometimes played instead by a notion of *non-factive ground* [Fine, 2012b, pp. 48-50].

One might suspect that the notions we have defined just come to the idea that $\leq$ only adds reflexive instances to $<$. This is not so: for there can be cases in which there is an explanatory argument from φ and auxiliary premises to ψ and another such argument in the reverse direction, for distinct φ and ψ. One plausible candidate is the above example of three objects of equal mass, where the "co-massiveness" of any two pairs explains the "co-massiveness" of the third pair. Another example, involving abstraction, was foreshadowed in Section 6 and will be developed in the next section. With some metaphysical ingenuity, yet further examples can be produced. Consider, say, a Lockean metaphysics of personal identity based on direct memory connections. Suppose a soldier S, fifty years ago, could remember performing the actions of a boy B, eighty years ago, and an old general G can now remember preforming the actions of S, fifty years ago.[40] Then, according to the Lockean view, $B = S$ and $S = G$ (each grounded in the mentioned memory connections) explain $B = G$. Yet by parity considerations, we should also have $B = G$ and $G = S$ explaining $B = S$. Of course, this account of personal identity has problems. But it illustrates how examples of the relevant form naturally arise.

If explanatory arguments are sometimes reversible in this way, what should we say about the correspondence between such arguments and (ordinary) grounding claims? Where there is an explanatory argument from φ and auxiliary premises to ψ and also vice versa, neither of these two facts can be placed strictly below the other. Here's the simplest fix. First, we define the idea that truths are "on the same level":

Coeval truths: A truth ϕ and a truth ψ are *coeval* iff there is an explanatory argument containing only truths from ϕ, Δ to ψ (for some Δ) and vice versa.[41]

Next we use that idea to define strict ground as weak ground that "descends" the hierarchy of explanatory levels:

Strict full ground: $\Delta < \phi$ iff $\Delta \leq \phi$ (*i.e.*, there is an explanatory argument containing only truths from Δ to ϕ) and no member of Δ is coeval with ϕ.[42]

So, assuming that the Lockean is correct about which inferences are explanatory, which are reversible, and which are not, we secure $B = S, S = G \not< B = G$ and $B = S, S = G \leq B = G$.

[40] This case is described by Reid [Reid, 2011, Essay III, Ch. 3] in one of his objections to Locke's [Locke, 1996, Bk. II, Ch. xxvii] view.

[41] This account may need to be tweaked in light of examples involving the ascription of truth and related notions. Consider, for instance, a disjunction D defined as $0 = 0 \vee True(D)$. Then, plausibly, there's an explanatory argument from $0 = 0$ to D, on to $True(D)$, and then on to D again. But one may still feel that $True(D)$ is strictly above D. The problems attending the specification of grounds for truth were first stated in [Fine, 2010]. See also [Correia, 2014, deRosset, 2021, Krämer, 2013, 2020, Litland, 2015, Lovett, 2020, Rodriguez-Pereyra, 2015, Woods, 2018] for discussion. These problems are orthogonal to our present concerns, since truth ascriptions are not on the scene.

[42] This, in essence, is the definition of $<$ in terms of $\leq$ stated in [Correia, 2010] and [Fine, 2012a], and endorsed and developed by [Lovett, 2020, pp. 2556-7].

Finally, with the notion of (sometimes reversible) explanatory arguments on the table, we now have a pleasingly simple and direct way to relate our (so far programmatic) notion of sufficiency to grounding and related notions. Namely: $\varphi \Rightarrow \psi$ can be understood as "there is an explanatory argument from φ to ψ." If we accept this link, we can hold on to RHS $\Rightarrow$ LHS for each instance of a permissible abstraction principle.

8 Application to ground-theoretic abstractionism

How does this help with the problem of autoabstraction? First, we need a specification of which inferences are explanatory. Corresponding to each of the grounding principles ($\vee <$)-($\rightarrow <$) listed in §5 and used in §6 to state the problem of autoabstraction is an introduction rule whose relevant instances must be explanatory if the grounding principle is true. The introduction rule corresponding to ($\rightarrow <$) is, for instance,

$$\boxed{\dfrac{\neg\phi}{(\phi \rightarrow \psi)} \qquad \dfrac{\psi}{(\phi \rightarrow \psi)}}$$

We preserve the spirit of these grounding principles by accepting that instances of the corresponding rules are explanatory. Similarly, we can replace commitment to ($\forall_{aa} <$) and ($\exists_{aa} <$) with the assumption that instances of the corresponding inference rules are explanatory. Similar remarks apply to ($\# <$), ($\neg\# <$), ($P <$), and ($\neg P <$). We also turn out to need natural specifications of explanatory inference rules for negations of conjunctions and disjunctions; a full list of explanatory inference rules can be found in Appendix A.[43] Finally, as we have already noted, we understand the sort of sufficiency expressed by $\Rightarrow$ as indicating the presence of an explanatory argument, rather than a conditional ground.

On these assumptions, it is straightforward to show that $1 = 1$ and $2 = 2$ are coeval. In fact, if $aa = 2$, then $2 = 2 \Rightarrow aa \approx aa \Rightarrow 1 = 1$, and if $bb = B, 1$ (where 'B' abbreviates 'Biden'), then $B = B, 1 = 1, B \neq 1, 1 \neq B \Rightarrow bb \approx bb \Rightarrow 2 = 2$. (The explanatory arguments establishing these results correspond to the steps used to state the autoabstraction problem in §6.) More generally,

Proposition 2 For any cardinal numbers n and m, $n = n$ and $m = m$ are coeval.

Thus, if we accept the conception of weak ground and the apparatus of explanatory arguments, the autoabstraction problem is transformed into a result: the self-identity of each of the cardinal

[43]The view we are developing is most naturally paired with a commitment to the claim that, except in special cases exemplified by the problem of autoabstraction, the fact expressed by a true conjunction is strictly grounded in the fact expressed by its conjuncts, that expressed by a true disjunction in those expressed by its true disjuncts, and $\neg\neg\phi$ in ϕ. The resulting view requires that we distinguish, in such cases, the fact expressed by ϕ from each of those expressed by $(\phi \wedge \phi), (\phi \vee \phi)$, and $\neg\neg\phi$. Notice that this relatively fine-grained approach is consistent with taking $\#xx$ to be purely referential, in the sense introduced in §4, so that if there are n members of xx, the fact expressed by, say, '$n \neq m$' is also expressed by '$\#xx \neq m$'.

numbers is coeval with the self-identity of each of the others. The structure of cardinal numbers is (strictly) grounded "all at once," if it is grounded at all. It is a consequence of this sort of abstractionism, then, that the cardinal numbers enjoy a kind of holistic interdependence that might be taken instead to provide intuitive motivation for mathematical structuralism [Linnebo, 2008, Litland, forthcoming].

A fully satisfactory solution to the problem, however, requires that we offer a systematic account of how that structure is (strictly) grounded. We get interestingly different results if we assume that we are given an empty plurality $\emptyset\emptyset$ than if we do not. So, suppose first that we are given $\emptyset\emptyset$. We have assumed that

$$\phi(a_1), \phi(a_2), \dots \Rightarrow (\forall x \in aa)\phi(x)$$

for all $aa = a_1, a_2, \dots$. A limit case of this assumption applied to $\emptyset\emptyset$ implies that $\emptyset \Rightarrow (\forall x \in \emptyset\emptyset)\phi(x)$. Since, vacuously, every sentence in $\emptyset$ is true, $(\forall x \in \emptyset\emptyset)\phi(x)$ is also true. Moreover, there is no sentence in $\emptyset$ coeval with $(\forall x \in \emptyset\emptyset)\phi(x)$. So, an immediate consequence of our assumption is that $(\forall x \in \emptyset\emptyset)\phi(x)$ is *zero-grounded*: $\emptyset < (\forall x \in \emptyset\emptyset)\phi(x)$. Similarly, $\emptyset < \neg(\exists x \in \emptyset\emptyset)\phi(x)$.

Moreover, 0 can play a key role in weakly grounding the ontology of positive cardinal numbers. First, $\emptyset\emptyset \approx \emptyset\emptyset$ is zero-grounded, and so $0 = 0$ is zero-grounded.[44] The non-existence of a relation of the relevant sort on $\emptyset\emptyset \times aa$ that is 1-1 and onto for any non-empty plurality aa is also zero-grounded,[45] as is the non-existence of such a relation on $aa \times \emptyset\emptyset$.[46] So $0 \neq n$ and $n \neq 0$ are each zero-grounded, for every positive n. If $aa = B$, then $B = B \Rightarrow 1 = 1$. A similar argument shows that $0 = 0 \Rightarrow 1 = 1$. The corresponding argument contains only true claims. Since $\emptyset \Rightarrow 0 = 0 \Rightarrow 1 = 1$, we have $\emptyset < 1 = 1$. When we move up to cardinals greater than 1, we get

(11) $0 = 0, 0 \neq 1, 1 \neq 0 \Rightarrow 2 = 2$.

Again, every sentence occurring in the corresponding explanatory argument is true. Since each of the truths on the LHS of (11) is zero-grounded, $2 = 2$ turns out to be zero-grounded. More generally, we can show:

Proposition 3 Suppose there is an empty plurality $\emptyset\emptyset$, i.e., $(\forall x)x \notin \emptyset\emptyset$. Then, for all $m, n \in \mathbb{N}$, where $m \neq n$, $\emptyset \Rightarrow m = m$ and $\emptyset \Rightarrow m \neq n$. Likewise, for all $m, n \in \mathbb{N}$, if $P(m, n)$ (or $\neg P(m, n)$), then $\emptyset \Rightarrow P(m, n)$ (or $\emptyset \Rightarrow \neg P(m, n)$).

[44] There is exactly one relation on $\emptyset\emptyset \times \emptyset\emptyset$, the empty relation $\boldsymbol{Z}$. Since the claim that $\boldsymbol{Z}$ is functional, total on $\emptyset\emptyset$, 1-1, and onto $\emptyset\emptyset$ is a conjunction of claims, each of the form $(\forall x \in \emptyset\emptyset)\phi$, it is zero-grounded.

[45] There is exactly one relation on $\emptyset\emptyset \times aa$, the empty relation $\boldsymbol{Z}$. Since aa is non-empty, $\neg(\exists b \in \emptyset\emptyset)\boldsymbol{Z}(b, a)$ is zero-grounded, for some $a \in aa$. So, $\boldsymbol{Z}$'s not being onto aa is zero-grounded. Thus, $\boldsymbol{Z}$'s not being functional, total on $\emptyset\emptyset$, 1-1, and onto aa is zero-grounded, and thus there being no such relation on $\emptyset\emptyset \times aa$ is zero-grounded.

[46] An argument similar to the previous case demonstrates that, if $aa = 0$, then $\emptyset \Rightarrow aa \not\approx \emptyset\emptyset$.

The proof, which is straightforward, is given in a supplement to this article.

In fact, we can extend Proposition 3 yet further and show that every truth of second-order Dedekind-Peano arithmetic is zero-grounded. See the technical supplement for a proof sketch. The argument outlined there crucially relies on the inference rules proposed for generalizations restricted to a plurality (in the present case, the plurality of all natural numbers).[47]

Finally, there is no problem of autoabstraction for $0 = 0$: no abstraction principle applies to any plurality containing any cardinal number to yield the self-identity of 0. Thus, if we allow the empty plurality $\emptyset\emptyset$, then our very natural assumptions entail that the entire ontology of natural numbers is zero-grounded. This is a very happy result—but it relies on there being an empty plurality.

9 Doing without the empty plurality

Such a commitment sounds implausible in ordinary English, since it comes to the claim that there are some things such that nothing is among them. But, we have suggested, there are legitimate languages that accept an empty plurality.[48] Some readers may disagree or wish to remain more faithful to ordinary English. Let us examine, therefore, whether we can avoid appealing to $\emptyset\emptyset$.

Our assumptions deliver an interesting result. As we have seen, we get $B = B \Rightarrow 1 = 1$. The corresponding argument has only true premises. It is safe to assume that there is no explanatory argument for Biden's self-identity that appeals to the self-identity of the cardinal number 1, and so no explanatory argument of that sort containing only true premises. So, we have $B = B < 1 = 1$. Though we have picked Biden, he is replaceable. Any other ordinary individual would serve as well. So, the self-identity (and existence) of 1 depends generically on the self-identity of every ordinary thing. When we move up to cardinals greater than 1, however, we get $B = B, B \neq 1, 1 \neq B \Rightarrow 2 = 2$. Every sentence occurring in the relevant explanatory argument is true. But whether we get a strict ground for $2 = 2$ turns on what explanatory arguments for the distinctness of Biden and 1 there are. If there is an explanatory argument containing only truths for $B \neq 1$ whose premises include $2 = 2$, then we have no corresponding strict ground. More generally, if we let $S_1 = \{\ulcorner B = B \urcorner\}$, and, for positive

[47]Our current project of grounding all the truths of second-order Peano arithmetic must not be conflated with that of deriving all the theorems of that system. The standard way to do the latter is to appeal to *Frege's theorem*, which states that Hume's Principle and second-order logic, coupled with Frege's definitions of the primitives of arithmetic, suffice to derive all the axioms of second-order Peano arithmetic [Boolos, 1990]. This result is available in our setting as well but needs a tiny tweak to accommodate the fact that we use a primitive predicate 'P' for numerical predecession, rather than Frege's defined predicate (cf. p. 8, fn. 17).

[48]Alternatively, our use of plural logic could be replaced by second-order logic restricted to properties that are "extensionally definite" (cf. fn. 53), since the empty property $\lambda x(x \neq x)$ is presumably "extensionally definite."

natural numbers n, let

$$S_{n+1} \quad = \quad \{\ulcorner B = B \urcorner \, , \, \ulcorner B \neq 1 \urcorner \, , \, \ulcorner 1 \neq B \urcorner \, , \, \ldots , \ulcorner B \neq n \urcorner \, , \, \ulcorner n \neq B \urcorner \}$$

then we can show:

Proposition 4 For all $m, n \in \mathbb{N}^+$, where $m < n$:

1. $S_m \Rightarrow m = m$;
2. $S_n \Rightarrow m \neq n$; and
3. $S_n \Rightarrow n \neq m$.

(See supplement.) So, in the absence of a systematic account of what grounds the non-identity of Biden with any of the natural numbers, we lack a systematic account of what grounds the arithmetical facts.

Our discussion, then, has a surprising upshot: if there is no empty plurality, then a fully satisfactory solution to the problem of autoabstraction along the lines we suggest requires that we confront a ground-theoretic version of Frege's famous *Caesar problem* [Frege, 1953, §56]. The original Caesar problem concerned the fact that Frege's proposed definition of number (deploying a property-theoretic version of HP) does not appear to determine whether Julius Caesar is a number. The ground-theoretic analogue, which we might call the *Biden problem*, is that our ground-theoretic gloss on HP provides us with no account of what makes Biden distinct from each of the numbers. In the absence of an account of how these distinctness facts are grounded, we have at best a partial account of how the arithmetical facts are grounded. What's more, we have no bottom-up grounding for the ontology of natural numbers by non-arithmetical facts, assuming that such facts as that Biden$\neq 1$ is (in part) arithmetical. So, we have made some progress, but there is still further to go.

It is not completely clear how to solve the Biden problem. But there is a plausible proposal in the offing. Let's consider distinctness claims involving cardinal numbers generally. Some such claims will be true and others will be false. For instance, it is overwhelmingly plausible that

(12) Biden$\neq 1$

is true. Before we get to that case, however, let's consider another. It is equally plausible that, if the pair set $\{a, b\}$ is the result of the application of the abstraction operation corresponding to

SA $\{aa\} = \{bb\}$ iff aa and bb are co-extensive[49]

then

(13) $\{a, b\} \neq 2$.

What grounds the distinctness of the set and the number? Plausibly, they owe their distinctness to the difference in the two forms of abstraction by which each is obtained. If this is correct, then the ground for their distinctness will contain the facts that license those abstractions. Thus, so long as neither a nor b are cardinal numbers or other entities abstractable from 2,

(14) a, b and a, b are co-extensive, $a, b \approx a, b < \{a, b\} \neq 2$.

That is, what makes the set and the number distinct is just the result of pooling the congeries of facts that ground the existence of each. To get a fully general account along these lines, we use $\Rightarrow$, with (strict) grounding defined in its terms, as in §7:

GD If $\Delta \Rightarrow x = x$, $\Gamma \Rightarrow y = y$, and $x \neq y$ then $\Delta, \Gamma \Rightarrow x \neq y$.

If we apply (GD) to (12), then we get the result that $B = B, 1 = 1 \Rightarrow B \neq 1$. Since, as we have seen, $B = B \Rightarrow 1 = 1$, we have

(15) $B = B \Rightarrow B \neq 1$.

Applying (GD) to $2 = 2$ we get

(16) $B = B, B \neq 1, 1 \neq B \Rightarrow 2 = 2$

Using (15), an obvious variant of (15), and the transitivity of $\Rightarrow$, we get

(17) $B = B \Rightarrow 2 = 2$ and $B = B \Rightarrow B \neq 2$.

Similar results obtain for the other finite cardinals. Assuming that no fact expressed by any arithmetical claim of either the form $n = n$ or $n \neq m$ is a weak ground for Biden's self-identity, this implies that the entire ontology of natural numbers is grounded in the self-identity of Biden. Biden is not special in this regard. Any human being or other ordinary individual, even the lowliest speck of dust, does the same grounding work.

Let's summarize. If we can swallow a commitment to the empty plurality, the zero-grounding of the entire ontology of natural numbers falls out of highly plausible principles concerning what explanatory arguments there are. On this view, that ontology is "thin" in a very strong sense: it is zero-grounded. God didn't have to do anything to make it the case

[49] (SA) appears to give rise to the bad company problem. See §10 below for discussion.

that there are numbers. If we can't swallow commitment to the empty plurality, then we get bottom-up grounding for that ontology only if we can solve the Biden problem, by saying what makes it the case that Biden is distinct from each of the natural numbers. We have tentatively suggested a solution to that problem, on which the entirety of arithmetic is grounded in the self-identity of any ordinary individual. On this view, God had only to create an individual – any individual whatsoever – to make it the case that there are numbers.[50]

10 The bad company problem

We claimed at the outset that ground-theoretic abstractionism opens the way to progress on the bad company problem. We can solve the problem by identifying a large and natural class of abstractions, all of which are permissible, and which includes key principles such as (HP). We propose to find such a class by generalizing the idea we have seen at work in the case of cardinal numbers, namely, that every true atomic statement (or negation thereof) about the abstracta can be grounded in facts that are intrinsic to the objects on which we abstract.

Let us first review the case of Hume's Principle. Let us continue to use $\approx$ to express (our formulation of) equinumerosity and '$\textsc{Prec}(xx, yy)$' to say that the result of removing one of yy bears $\approx$ to xx. Then:

$$\begin{array}{cc} xx \approx yy \Rightarrow \#xx = \#yy & xx \not\approx yy \Rightarrow \#xx \neq \#yy \\ \textsc{Prec}(xx, yy) \Rightarrow P(\#xx, \#yy) & \neg\textsc{Prec}(xx, yy) \Rightarrow \neg P(\#xx, \#yy) \end{array}$$

On the left-hand side of the arrows '$\Rightarrow$', which record explanatory arguments, we have statements that are intrinsic to a domain of objects assumed to be in good standing. Let us call these the "old" objects. On the right-hand side, we find statements about certain objects obtained by abstraction on zero or more "old" objects. Some of these abstracta may themselves be "old" objects, as the phenomenon of autoabstraction shows; but if they are not, we say that they are "new". On the analysis we have provided of the statement that xx are equinumerous with yy, we have shown that the statements that figure on the right are (if true) weakly grounded in a collection of facts that patently involve only xx and yy, namely, identity and distinctness facts concerned solely with members of these two pluralities, respectively. Thus, all atomic and negated atomic truths about any "new" abstracta can be derived, using explanatory arguments, from truths expressing facts that are intrinsic to the "old" objects on which we abstract.

It is important to notice that the distinction between "old" and "new" does not correspond

[50] Donaldson [2020] expresses a favorable view of a proposal broadly along these lines as a response to the challenges posed in his Donaldson [2017].

to properties of the relevant objects, but is only defined relative to an explanatory argument. There can be alternative explanatory arguments that reverse the roles of "old" and "new". Relative to some explanatory arguments, we observed, 1 is an "old" object on the way to 2, whereas relative to others, 2 is an "old" object from which 1 is abstracted. What matters for securing the permissibility of abstraction is the availability of a strictly bottom-up route to the relevant abstracta: a route every step of which uses objects already explained to explain the existence and properties of further objects.

Consider now the analogous explanatory arguments associated with the plural version of Basic Law V:

$$xx \equiv yy \Rightarrow \{xx\} = \{yy\} \qquad xx \not\equiv yy \Rightarrow \{xx\} \neq \{yy\}$$
$$x \in yy \Rightarrow x \in \{yy\} \qquad x \notin yy \Rightarrow x \notin \{yy\}$$

where $xx \equiv yy$ abbreviates $(\forall x \in xx)(x \in yy) \wedge (\forall y \in yy)(y \in xx)$. Notice that we have the same desirable (weak) grounding of atomic and negated atomic facts about the "new" abstracta in facts that are intrinsic to the "old" objects on which we abstract. For example, given our analysis of plurality-restricted quantifiers, such as '$\forall x \in xx$', the statement '$xx \equiv yy$' is intrinsic to xx and yy.

Our proposal may seem doomed, however. Don't we already know that Basic Law V—in its plural, just as well as its customary second-order formulation—is inconsistent? But this inconsistency makes non-trivial demands on the plural logic. Our response is to ensure that the pluralities xx and yy on which we abstract are *available for abstraction*, in the sense that the relevant objects all exist at some stage of the grounding hierarchy where the existence of their sets can then be grounded.[51] We explain how this can be ensured in Appendix B. For now, we simply assume that the values assigned to any plural variable are all available at some stage or other.

It is instructive to contrast the plural version of Basic Law V with the second-order version, where we abstract, not on pluralities, but on properties. Suppose we adopted the following explanatory arguments:

$$F \equiv G \Rightarrow \{u : Fu\} = \{u : Gu\} \qquad F \not\equiv G \Rightarrow \{u : Fu\} \neq \{u : Gu\}$$
$$Fx \Rightarrow x \in \{u : Fu\} \qquad \neg Fx \Rightarrow x \notin \{u : Fu\}$$

These principles differ in an important way from their plural analogues. Where a plurality is

[51] Similar responses are developed by [Florio and Linnebo, 2020] and [Florio and Linnebo, 2021, ch. 12] (in the special case of set abstraction) and [Kment, 2022] (in the context of the Russell-Myhill paradox).

delimited with respect to its members, a property need not be delimited with respect to its instances. When we are "given" a plurality, we are given all of its members, once and for all. This is why '$xx \equiv yy$' is intrinsic to the objects in question. By contrast, we can be given a property without being given all, or indeed any, of its instances. Consider, as in Section 5, the property of being a star. To determine its instances, we need to consider all of reality. This means that the statements found on the left-hand side of the above explanatory arguments require, for their grounding, unbounded quantification and thus cannot be seen as intrinsic to the "old" objects on which we abstract.

This unbounded quantification is potentially problematic, since it removes the assurance we had, in the analogous plural abstraction, of the availability of a strictly bottom-up route to the relevant abstracta. To show that a problem actually arises, let $F = \lambda x(x \in x)$ and $f = \{u : Fu\}$. It is reasonable to assume $\varphi(a) \Rightarrow \lambda x.\varphi(x)(a)$ and $\neg\varphi(a) \Rightarrow \neg\lambda x.\varphi(x)(a)$. Together with the abstract-based explanatory arguments stated above, this yields:

$$f \in f \;\Rightarrow\; Ff \;\Rightarrow\; f \in f$$

$$f \notin f \;\Rightarrow \neg\, Ff \;\Rightarrow\; f \notin f$$

Thus, we have an atomic or negated atomic truth concerned with f, but the explanatory inference rules we have specified provide no stricly bottom-up explanatory route to that truth. Nor is there any other explanatory route to the relevant truth.[52] Nor, for that matter, would it help to restrict to "available" properties, since a property can be "available" even though its instances are not.[53]

In sum, there is a huge difference between the plural and the ordinary second-order versions of Basic Law V, at least on the assumption that every plurality is available at some stage or other: where the former seems acceptable because it grounds all atomic and negated atomic facts about the "new" abstracta in facts intrinsic to the "old" objects on which we abstract, the latter is unacceptable because some of its instances foreclose the possibility of any such grounding.

We wish to generalize this insight. Consider an equivalence relation $\sim$ on pluralities on

[52]Since we assume the "old" objects to be in good standing, it follows that f is "new".

[53]Let us say that a collection C is *extensionally definite* iff $(\forall x \in C)\varphi(x)$ is weakly grounded in its critical instances, that is, in the instances $\varphi(a)$ for every $a \in C$. We have suggested (as against [Skiles, 2015, Donaldson, 2017]) that, while every plurality is extensionally definite, properties in general are not. Some special properties, however, *are* extensionally definite, such as the property of being one of aa or of being an x such that $\bigvee_{\gamma\in\Gamma}(x = a_\gamma)$. Thus, the work to which we put plurals could instead be done by second-order logic restricted to extensionally definite properties. Notice in particular that the empty property is extensionally definite.

which we would like to abstract:

$$\text{(AP)} \qquad \S xx = \S yy \leftrightarrow xx \sim yy$$

Next, let us generalize what we did to obtain the predecessor relation among cardinal numbers and the membership relation for sets. To this end, suppose $\sim$ is a congruence with respect to $\varphi(xx, yy)$, that is:

$$xx \sim xx' \wedge yy \sim yy' \ \rightarrow \ (\varphi(xx, yy) \leftrightarrow \varphi(xx', yy'))$$

Since φ does not distinguish between $\sim$-equivalent pluralities, we can try to define a relation φ^* among the abstracta that is "inherited" from the obtaining of φ among the pluralities on which we abstract:

$$\text{(Inher)} \qquad \varphi^*(\S xx, \S yy) \leftrightarrow \varphi(xx, yy)$$

Analogous principles can be formulated for formulas with a different number of variables.

So far, so familiar. Let us now propose a crucial restriction. Say that a formula φ of plural logic, with at least one free plural variable, is *plurally bounded* just in case each of its quantifiers is restricted to one of its free plural variables.[54] (For example, the statement that xx and yy are coextensive is plurally bounded.) We extend the definition to formulas of generalized plural logic by additionally admitting quantifiers restricted to any generalized plurality based on the values of the formula's free plural variables.[55] (For example, in Section 8 and Appendix A we formulated the plural version of (HP) using quantification over higher pluralities that relate xx and yy.) The crucial restriction is that $xx \sim yy$ and $\varphi(xx, yy)$ be *plurally bounded* and use only atomic predicates that stand for intrinsic relations. When this requirement is met, we contend, there is an acceptable form of abstraction associated with the following explanatory

[54] Formally, a quantifier $\forall u$ (or $\forall uu$) is restricted to vv iff each occurrence has the form $\forall u(u \in vv \rightarrow \ldots)$ (or $\forall uu(uu \subseteq vv \rightarrow \ldots))$, as are our primitive restricted quantifiers, $(\forall x \in vv)$, and their plural analogues.

[55] Generalized pluralities are explained and defended in [Florio and Linnebo, 2021, ch. 9], whose Appendix 9.A provides a precise explication of the intuitive notion of being "based on".

arguments:[56]

$$xx \sim yy \Rightarrow \S xx = \S yy \qquad xx \not\sim yy \Rightarrow \S xx \neq \S yy$$

$$\varphi(xx, yy) \Rightarrow \varphi^*(\S xx, \S yy) \qquad \neg\varphi(xx, yy) \Rightarrow \neg\varphi^*(\S xx, \S yy)$$

The reason, yet again, is that there are instances of explanatory arguments that ensure that every true atomic or negated atomic fact about the "new" abstracta is (weakly) grounded in facts intrinsic to the "old" objects on which we abstract. We thereby secure the possibility of a strictly bottom-up route to the relevant abstracta.

Just as desired, this yields a large and natural class of permissible abstractions that includes both (HP) and plural set abstraction.[57] Our informal argument is borne out by a formal consistency proof in Appendix C.

11 Concluding summary

We set out to develop a version of abstractionism using the notion of metaphysical grounding. This aim is important, we argued, because it promises to solve the two main challenges to abstractionism in one go. We believe our two-pronged aim has been achieved.

First, we have proposed to understand the claim that the abstracta are thin relative to the objects on which we abstract in terms of the grounding of atomic and negated atomic facts about the former in facts that are intrinsic to the latter. (As announced, our intuitive notion of intrinsicness receives a precise, technical explication in Appendix C.) So, grounded abstraction yields thin abstracta.

Second, we have just seen that by enforcing this grounding requirement, we also ensure that the abstraction is permissible: we obtain a systematic and well-motivated response to the bad company problem, which counts a large and mathematically important class of abstractions as permissible. Thus, grounded abstraction ensures permissible abstraction.

A side benefit of our investigation is to demonstrate that the theory of grounding is capable

[56]This way of developing abstractionism will encounter problems when the equivalence relation $\sim$ can hold contingently. For example, on an abstractionist view of shapes, a certain shape, S, exists and is the shape of a in virtue of the fact that a is geometrically similar to itself. But it is possible that a have had a different shape, while still being self-similar. So, it seems, the actual grounds for the existence of S fail to necessitate its existence. This problem, due to [Donaldson, unpublished], is difficult and cannot be solved here. Pending a better solution, we can either abstain from grounding necessitarianism or else restrict to cases in which the relevant equivalence relation $xx \sim yy$ holds rigidly if at all. (The former is possible because the consistency proof in Appendix C, while motivated by our account of grounded abstraction, does not explicitly involve grounding or non-actual possibilities.)

[57]To be clear: we are not saying that abstraction on properties or relations that are not extensionally definite is always impermissible, only that it poses a special challenge, which will require an answer different from the one we develop here. Our claim to have identified one large and natural class of permissible abstractions is entirely consistent with the existence of other such classes.

of doing valuable explanatory work. This provides ammunition against skeptics concerning grounding. We believe our articulation and explanatory use of the less familiar notion of weak grounding is particularly significant.

Appendices

A Explanatory inference rules and equinumerosity

As is standard, we take $aa \approx bb$ to mean that one among the relations on $aa \times bb$ is of an appropriate sort.[58] What sort of relation is appropriate? Certainly the relation will need to be functional, total on aa, 1-1, and onto bb. But additionally, to avoid the consequence that any fact whatsoever is among the grounds of arithmetical truths, the witnessing relation must be a relation-in-extension, rather than a relation-in-intension.[59] After all, a bijective relation between aa and bb may take the form *being an x and y such that ϕ and (($x = a_1$ and $y = b_1$) or ($x = a_2$ and $y = b_2$) or ...* (for $(a_i) = aa$ and $(b_i) = bb$, and any arbitrary truth ϕ). It is then plausible to think that ϕ will be among the partial grounds for the claim that there is a bijective relation-in-intension between aa and bb, *i.e.*, that $aa \approx bb$. This is obviously contrary to the spirit of the abstractionist view.

Instead, we will assume that, for any pluralities aa and bb, there is also a higher-order plurality $aa \otimes bb$ of the binary relations-in-extension on $aa \times bb$.[60] Instances of non-empty members f of $aa \otimes bb$ will be grounded in accordance with their definitions and the ordinary principles of grounding. So, for instance, $f(a, b)$ is grounded in any of the ways that

$$((a = a_1 \wedge b = b_1) \vee (a = a_2 \wedge b = b_2) \vee \ldots)$$

is. On this view, we have no reason to think $\phi \prec f(a, b)$, for arbitrary truths ϕ.

This leaves the question of how to deal with the empty binary relation $\boldsymbol{Z}$ on $aa \otimes bb$. This relation never holds, so there is no problem concerning how $\boldsymbol{Z}(a, b)$ is grounded. But we do face the question of how $\neg\boldsymbol{Z}(a, b)$ is grounded for $a \in aa$ and $b \in bb$, and there is no

[58] By a 'relation on $aa \times bb$' we mean a relation whose domain is a sub-plurality of aa and whose range is a sub-plurality of bb.

[59] By 'relations-in-extension' we mean relations that can be defined (if need be, using infinitary resources) directly in terms of their graphs, as opposed to being defined in graph-independent terms that (contingently or even necessarily) determine that graph. There is, for example, a dyadic relation-in-extension defined by relating a_1 to b_1, a_2 to b_2, etc.

[60] We might instead characterize $aa \otimes bb$, where $aa = a_1, a_2, \ldots$ and $bb = b_1, b_2, \ldots$, as a pluplurality of pluralities of ordered pairs $\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \ldots$ that are, in effect, relations-in-extension on $aa \times bb$. The approach in the main text is simpler, since it does not require us to concern ourselves with questions concerning grounds for the relevant claims regarding ordered pairs. It is plausible, however, to assume that something similar to what we assume regarding grounds for the fact that $f(a, b)$, for some relation-in-extension f, holds for grounding the fact that $\langle a, b\rangle \in ff$ for some plurality of ordered pairs ff.

disjunction listing the instances to which we might appeal. The simplest and most intuitive view is that the self-identities of a and b jointly make it the case that they are not so related: $a = a, b = b < \neg \boldsymbol{Z}(a, b)$.[61]

We can now appeal to relations-in-extension to interpret $aa \approx bb$ to mean that some relation-in-extension in $aa \otimes bb$ is functional, total on aa, 1-1, and onto bb:

$(\exists f \in aa \otimes bb)($

functional: $(\forall b_1 \in bb)(\forall b_2 \in bb)(\forall a \in aa)(f(a, b_1) \wedge f(a, b_2) \rightarrow b_1 = b_2) \wedge$

total: $(\forall a \in aa)(\exists b \in bb) f(a, b) \wedge$

1-1: $(\forall a_1 \in aa)(\forall a_2 \in aa)(\forall b \in bb)(f(a_1) = b \wedge f(a_2) = b \rightarrow a_1 = a_2) \wedge$

onto: $(\forall b \in bb)(\exists a \in aa) f(a, b)$

$)$

This interpretation of equinumerosity claims tells us what grounds the arithmetical facts, given a suitable stock of explanatory arguments. To that end, suppose that each instance of the following rules is explanatory.

Truth functions:

$$\frac{\phi}{(\phi \vee \psi)} \qquad \frac{\psi}{(\phi \vee \psi)} \qquad\qquad \frac{\phi \qquad \psi}{(\phi \wedge \psi)} \qquad\qquad \frac{\neg\phi}{(\phi \rightarrow \psi)} \qquad \frac{\psi}{(\phi \rightarrow \psi)}$$

$$\frac{\neg\phi, \neg\psi, \ldots}{\neg(\phi \vee \psi \vee \ldots)} \qquad\qquad \frac{\neg\phi}{\neg(\phi \wedge \psi \wedge \ldots)} \qquad \frac{\neg\psi}{\neg(\phi \wedge \psi \wedge \ldots)} \qquad \frac{\ldots}{\ldots}$$

Plural membership: For each a and each plurality $aa = a_1, a_2, \ldots$:

$$\frac{a = a_i}{a \in aa} \qquad\qquad \frac{a \neq a_1 \quad a \neq a_2 \quad \ldots}{a \notin aa} \qquad\qquad \frac{a = a}{a \notin \emptyset\emptyset}$$

Quantifiers: For each plurality $aa = a_1, a_2, \ldots$:

[61] This would be the upshot if we were to characterize relations-in-extension on $aa \times bb$ not just in terms of their instances, but also in terms of their non-instances among $aa \times bb$. In general, this proposal would characterize R as if it had the form *being an x and y s.t. both ((${x = a_1}$ and $y = b_1$) or ($x = a_2$ and $y = b_2$) or ...) and $\neg$(($x = a'_1$ and $y = b'_1$) or ($x = a'_2$ and $y = b'_2$) or ...)*. This characterization specifies not just an extension, but also an anti-extension for each relation in $aa \otimes bb$. In the classical setting which is appropriate here the specification of an anti-extension is redundant for non-empty relations, but it would be needed in a more general setting. In any case, if we include both extensions and anti-extensions, then $\boldsymbol{Z}$ is not a special case, but is instead the relation-in-extension on $aa \times bb$ with null extension and full anti-extension.

$$\frac{\phi(a_1) \quad \phi(a_2) \quad \ldots}{(\forall x \in aa)\phi(x)} \qquad \frac{\phi(a_i)}{(\exists x \in aa)\phi(x)}$$

$$\frac{\neg\phi(a_1) \quad \neg\phi(a_2) \quad \ldots}{\neg(\exists x \in aa)\phi(x)} \qquad \frac{\neg\phi(a_i)}{\neg(\forall x \in aa)\phi(x)}$$

For pluralities $aa, aa_1, aa_2, \ldots$, where $aa_1, aa_2, \ldots$ are exactly the subpluralities of aa:

$$\frac{\phi(aa_1) \quad \phi(aa_2) \quad \ldots}{(\forall xx \subseteq aa)\phi(xx)} \qquad \frac{\phi(aa_i)}{(\exists xx \subseteq aa)\phi(xx)}$$

$$\frac{\neg\phi(aa_1) \quad \neg\phi(aa_2) \quad \ldots}{\neg(\exists xx \subseteq aa)\phi(xx)} \qquad \frac{\neg\phi(aa_i)}{\neg(\forall xx \subseteq aa)\phi(xx)}$$

Arithmetic: Let $(\phi \leftrightarrow \psi)$ abbreviate $(\phi \to \psi) \wedge (\psi \to \phi)$ and $\textsc{Prec}(xx, yy)$ abbreviate

$$(\exists yy' \subseteq yy)(\exists y \in yy)(\ (\forall z \in yy)(z \in yy' \leftrightarrow z \neq y)\ \wedge\ xx \approx yy\ ).$$

For aa, bb, m, n such that m is the number of aa and n the number of bb:

$$\frac{aa \approx aa}{m = m} \qquad \frac{aa \not\approx bb}{m \neq n} \qquad \frac{\textsc{Prec}(aa, bb)}{P(m, n)} \qquad \frac{\neg\textsc{Prec}(aa, bb)}{\neg P(m, n)}$$

Here negated disjunctions and conjunctions are treated as if they can be inferred in explanatory arguments from the same premises as their unnegated DeMorgan equivalents. Similar remarks apply to negated restricted existential and universal quantifications.

Now we can rigorously define the notion of an explanatory argument corresponding to a weak grounding claim. A *derivation* of the formula ϕ from the set of formulas Δ is a sequence of formulas $\phi_1, \phi_2, \ldots, \phi_n$, where $\phi_n = \phi$ and ϕ_k, for each $k = 1, 2, \ldots, n$, is either a member of Δ or follows from preceding formulas in the sequence by one of the rules specified above. Given a derivation $\phi_1, \phi_2, \ldots, \phi_n$, say that ϕ_k *figures as a premise* if, for some $m > k$, ϕ_k/ϕ_m is an instance of one of the above rules or if, for some $m > k$ and $l_1, \ldots, l_n < m$, $\phi_k, \phi_{l_1}, \ldots, \phi_{l_n}/\phi_m$ is an instance of one of the above rules. A derivation $\phi_1, \phi_2, ..., \phi_n = \phi$ is said to be *relevant* when each non-terminal formula ϕ_k for $k < n$ figures as a premise in the derivation.[62] We write $\Delta \Rightarrow \phi$ to indicate the existence of a relevant derivation of ϕ from Δ, and we say that the derivation *corresponds to* the weak grounding claim $\Delta \leq \phi$.

By construction, $\phi \Rightarrow \phi$, and it is easy to show ([deRosset and Fine, 2023, T3.6]) that $\Rightarrow$ is closed under CUT; that is, if $(\phi_i) \Rightarrow \phi$ and $(\Delta_i \Rightarrow \phi_i)$, then $(\Delta_i) \Rightarrow \phi$. As a consequence, $\Rightarrow$ is

[62]This specification of the notion of a derivation is taken from [deRosset and Fine, 2023, §3].

also closed under AMALGAMATION; that is, if $(\Delta_i \Rightarrow \phi)$, then $(\Delta_i) \Rightarrow \phi$. On the conception of weak ground developed in §7, $\Delta \leq \phi$ iff there is a relevant derivation of ϕ from Δ containing only truths.

Our suggestions for interpreting $f(a,b)$ then imply that $a = a_i, b = b_i \Rightarrow f(a,b)$ (for each i), and $\phi_1, \phi_2, \cdots \Rightarrow \neg f(a,b)$, where, for each i ϕ_i is either $a \neq a_i$ or $b \neq b_i$. Finally, in the case of the empty relation $\boldsymbol{Z}$ on $aa \times bb$, we have, for each $a \in aa, b \in bb$, $a = a, b = b \Rightarrow \neg \boldsymbol{Z}(a,b)$. Thus, we have a view concerning what $aa \approx bb$ says and how it is grounded that is consistent with the idea that $aa \approx bb$ is intrinsic to aa and bb. The claims in §§8,9 concerning sufficiency ($\Rightarrow$) claims for the arithmetical truths then follow. See the technical supplement to this article for details.

B Critical plural logic

As explained, we need a plural logic which ensures that each plurality is available at some stage of the grounding hierarchy, where the existence of the associated set can then be grounded. This requirement conflicts with the unrestricted plural comprehension scheme of traditional plural logic:

$$\exists y\, \varphi(y) \to \exists xx \forall y (y \in xx \leftrightarrow \varphi(y))$$

For the condition '$y = y$' would give rise to a universal plurality, encompassing every object whatsoever. But no such plurality is available at any single stage of the hierarchy.

Instead, we can use the *Critical Plural Logic* formulated and defended in [Florio and Linnebo, 2020] and [Florio and Linnebo, 2021, ch. 12]. This logic is obtained from traditional plural logic by replacing the unrestricted plural comprehension scheme with the following axioms:

- *Singletons*: For every object, there exists a plurality whose sole member is that object.
- *Adjunction*: For every plurality xx and object y, there exists a plurality whose members are precisely xx and y.
- *Generalized Union*: For every plurality each of whose members serves as a "tag" on a plurality of objects, there exists a plurality whose members include all and only objects that occurs as a member of one of the "tagged" pluralities. More precisely, suppose there are xx such that:

  $$\forall x (x \in xx \to \exists yy \forall z (z \in yy \leftrightarrow \psi(x,z)))$$

  (That is, xx serve as "tags" and '$\psi(x,z)$' states that the "tag" x is applied to z.) Then there are zz such that:

$$\forall y(y \in zz \leftrightarrow \exists x\,(x \in xx \;\wedge\; \psi(x,y)))$$

- *Infinity*: For every formula $\psi(x,y)$ defining a functional relation and every plurality xx, there exists a plurality yy containing xx and closed under the relation defined by $\psi(x,y)$:

$$\forall x \exists! y\, \psi(x,y) \rightarrow \forall xx \exists yy(xx \preccurlyeq yy \,\wedge\, \forall x \forall y(x \in yy \wedge \psi(x,y) \rightarrow y \in yy))$$

  where $xx \preccurlyeq yy$ abbreviates $\forall z(z \in xx \rightarrow z \in yy)$.

These axioms entail plural analogues of the familiar set-theoretic axioms of Separation and Replacement as well; see the mentioned works for details.

Our decision to use critical, rather than traditional, plural logic can be understood in two different ways. On the *moderate view*, traditional plural logic is valid but, when studying grounded abstraction, it is expedient to focus on a special class of bounded pluralities, which we axiomatize by means of the critical logic. On the *radical view*, traditional plural logic is invalid and should be replaced by its critical cousin.

Let us consider what is at stake. A pleasingly simple version of our theory of abstraction says that it is permissible to abstract on any plurality, which would then play the role of "old" objects, provided that all atomic and negated atomic facts about the ensuing abtracta are (weakly) grounded in facts intrinsic to these "old" objects. When this simple theory is combined with our view that every plurality is extensionally definite (see fn. 53), it follows (as explained on pp. 27ff.) that set abstraction is permissible on *any* plurality. And this, in turn, means that every plurality is bounded at some stage of the hierarchy—namely, by the stage where the associated set is formed—and that traditional plural logic therefore must be restricted.

Thus, we face a choice. The radical option is to accept the simple theory of permissible abstraction, and therefore conclude we must reject traditional plural logic in favor of critical. The moderate option is to retain traditional plural logic and restrict the account of permissible abstraction by requiring, not only that atomic and negated atomic facts about the "new" objects be (weakly) grounded in facts intrinsic to the "old", but also that the pluralities on which we abstract be bounded in the grounding hierarchy. Thankfully, we need not resolve the matter here, because radicals and moderates alike will agree that our project calls for Critical Plural Logic.

C Consistency of plurality-bounded abstraction

Let us verify our claim that any form of abstraction on pluralities is permissible provided the equivalence and predicates to be inherited by the abstracta use only atomic predicates standing for intrinsic relations and are plurally bounded. (A formula is plurally bounded, we recall from p. 29, iff each of its quantifiers is restricted to one of its free plural variables.) We do this by providing a set-theoretic consistency proof.

Let $\mathcal{L}$ be any plural language and $\mathcal{M}$ be a set-theoretic $\mathcal{L}$-model, where plural variables are interpreted as ranging over (non-empty) subsets of the first-order domain. Let us write $[\![\varphi]\!]_{\mathcal{M}}$ for the extension of an open formula φ in $\mathcal{M}$. For example, if φ has a single free variable 'x', then $[\![\varphi]\!]_{\mathcal{M}} = \{a \in M : \mathcal{M} \models \varphi[a/x]\}$ (where, for readability, we suppress variable assignments in favor of the objects that they assign to the displayed variables). More generally, if φ has m free singular and n free plural variables, this extension is a subset of $M^m \times \wp(M)^n$. We say that one such model, $\mathcal{N}$, is a *permissible extension* of another, $\mathcal{M}$, (in symbols: $\mathcal{N} \sqsupseteq \mathcal{M}$) just in case:

(i) $\mathcal{N}$ extends $\mathcal{M}$;

(ii) for every atomic predicate P of $\mathcal{L}$, we have

$$[\![P]\!]_{\mathcal{N}} \cap (M^m \times \wp(M)^n) = [\![P]\!]_{\mathcal{M}}$$

where P has m singular argument places, followed by n plural ones.

We may think of (ii) as requiring that every atomic predicate express a relation that is intrinsic, in the sense that whether or not it obtains among some members of $\mathcal{M}$ is unaffected by the addition of further objects and further facts.

This suggests a precise, technical explication of our intuitive notion of intrinsicness.

Definition 1 A formula φ is *absolute with respect to some objects xx* iff for every model $\mathcal{M}$ containing xx and every permissible extension $\mathcal{N}$ of $\mathcal{M}$, we have: $[\![\varphi]\!]_{\mathcal{N}} \cap (M^m \times \wp(M)^n) = [\![\varphi]\!]_{\mathcal{M}}$.

Lemma 1 Any plurally bounded open formula is absolute with respect to the union of the pluralities assigned to its free plural variables.

Proof. An easy induction on the syntactic complexity of the formula. The case of atomic formulas is ensured by (ii) above. Truth-functional combinations of absolute formulas are in turn absolute. Finally, quantifiers restricted to objects available already in $\mathcal{M}$ contribute in the same way in $\mathcal{M}$ and in any $\mathcal{N} \sqsupseteq \mathcal{M}$ and thus preserve absoluteness. $\dashv$

Let $\mathcal{L}^+$ be the language that results from adding to $\mathcal{L}$ a singular term-forming operator §, applicable to plural terms, and a new predicate φ^*, with two singular argument places.

Proposition 5 Let $xx \sim yy$ be an equivalence relation that is definable in $\mathcal{L}$. Suppose this equivalence is a congruence with respect to the $\mathcal{L}$-formula $\varphi(xx, yy)$. Suppose further that $xx \sim yy$ and $\varphi(xx, yy)$ are absolute with respect to the union of xx and yy. Then the $\mathcal{L}^+$-theory consisting of

(AP) $$\S xx = \S yy \leftrightarrow xx \sim yy$$

(Inher) $$\varphi^*(\S xx, \S yy) \leftrightarrow \varphi(xx, yy)$$

is consistent in the context of Critical Plural Logic.

Proof. We proceed by building up a sequence of partial $\mathcal{L}^+$-models, where the abstraction operator § is only partially defined. The members of this sequence represent the results of undertaking more and more rounds of abstraction on all of the previously available pluralities. The union of these partial models is an ordinary, complete $\mathcal{L}^+$-model with a complete interpretation of §.

We can start with any $\mathcal{L}$-model, whose domain M_0 can be assumed to contain only non-sets. The plural variables range over $\wp^+(M_0)$, where $\wp^+(X)$ is defined as the set of non-empty subsets of the set X. (It would be straightforward to adapt our construction to the case where we admit an empty plurality and thus use the full powerset operation.) Interpreting the predicate φ^* as having the empty extension, we obtain a partial $\mathcal{L}^+$-model, which we designate $\mathcal{M}_0$.

Next, we wish to represent the abstract $\S xx$, for each plurality xx of elements of M_0. To this end, we consider $M_0 \cup \wp^+(M_0)$, using the set $\{xx\}$ to represent $\S xx$. Since we wish to identify equivalent representations of one and the same abstract, we define an equivalence relation $\simeq_1$ on $M_0 \cup \wp^+(M_0)$ as follows. First, members of M_0 are equivalent iff they are identical. Second, $\{xx\}$ and $\{yy\}$ are equivalent iff $xx \sim yy$ in $\mathcal{M}_0$. Third, no other equivalences obtain; in particular, no element x of M_0 is equivalent with a set $\{xx\}$. The first-order domain available after one round of abstraction is represented by the quotient $(M_0 \cup \wp^+(M_0))/{\simeq_1}$, which we designate N_1. Notice, however, that there is a natural injection i_0 of M_0 into N_1 defined by mapping x to its own equivalence class. It is expedient to define the domain M_1 after one round of abstraction as $M_0 \cup (N_1 \setminus i_0[M_0])$, where $i_0[M_0]$ is the image of M_0 under i_0. (Intuitively, we retain all the "old" objects in M_0 and add unique representatives of all the "new" abstracts.)

We now define the associated model $\mathcal{M}_1$. For any xx from M_0, we let $\S xx$ be the equivalence class of $\{xx\}$. As before, we let the plural variables range over $\wp^+(M_1)$. For any non-logical predicate of $\mathcal{L}$, we let its extension in $\mathcal{M}_1$ be identical with its extension in $\mathcal{M}_0$, in keeping with the requirement that these predicates stand for intrinsic relations. Finally, we let the extension of φ^* on M_1 be determined by φ in $\mathcal{M}_0$; that is, for xx and yy from M_0, we let φ^* apply to the equivalence classes of $\{xx\}$ and $\{yy\}$, in that order, iff $\varphi(xx, yy)$ in $\mathcal{M}_0$. (This is well-defined because $\sim$ is a congruence with respect to φ.) This provides our partial $\mathcal{L}^+$-model $\mathcal{M}_1$, where § is defined on all and only pluralities from M_0.

Two observations are in order. First, it is easy to see that $\mathcal{M}_1 \sqsupseteq \mathcal{M}_0$. Second, our absoluteness assumptions concerning $\sim$ and φ ensure that every instance of (AP) and (Inher) with parameters from M_0 is true in $\mathcal{M}_1$. To prove this, consider any xx and yy from M_0. Then: $\S xx = \S yy$ in $\mathcal{M}_1$ iff (by definition) $xx \sim yy$ in $\mathcal{M}_0$, iff (by Lemma 1) $xx \sim yy$ in $\mathcal{M}_1$. The case of (Inher) is analogous.

For all later successor stages, we proceed in a similar way. Suppose we have constructed $\mathcal{M}_\gamma$. We define an equivalence $\simeq_{\gamma+1}$ on $M_\gamma \cup \wp^+(M_\gamma)$ that equates representations of abstracts in accordance with $\sim$. More precisely, the first two clauses of the definition of $\simeq_{\gamma+1}$ parallel those used above. For the third clause, though, we equate $\S xx$ (from M_γ) with $\{yy\}$ (from $\wp^+(M_\gamma)$) iff $xx \sim yy$. This clause handles the phenomenon of autoabstraction. The clause is well-defined because it is independent of the choice of xx to represent the equivalence class $\S xx$. The rest of the construction proceeds as in the case of $\mathcal{M}_1$. We define $N_{\gamma+1}$ as the set of $\simeq_{\gamma+1}$-equivalence classes of $M_\gamma \cup \wp^+(M_\gamma)$, observe that there is a natural injection i_γ of M_γ into $N_{\gamma+1}$, and let $M_{\gamma+1}$ be $M_\gamma \cup (N_{\gamma+1} \setminus i_\gamma[M_\gamma])$. We let plural variables range over non-empty subsets of $M_{\gamma+1}$ and define predicates on this model in the same way as before. Finally, to interpret $\S xx$, for some xx from M_γ, we consider the equivalence class e of $\{xx\}$ in $N_{\gamma+1}$. If e is the i_γ-image of some $d \in M_\gamma$, we let $\S xx$ be d; otherwise, we let $\S xx$ be e. (Intuitively, if the abstraction yields an "old" object, present already in M_γ, we let $\S xx$ be this object; otherwise, we let $\S xx$ be the "new" representative of this abstract in $N_{\gamma+1}$.)

The result, once again, is a partial $\mathcal{L}^+$-model $\mathcal{M}_{\gamma+1} \sqsupseteq \mathcal{M}_\gamma$, where § is defined on all and only pluralities from M_γ, and where every instance of (AP) and (Inher) with parameters from M_γ is true in $\mathcal{M}_{\gamma+1}$.

At any limit stage λ, we take unions. Thus, for first-order domains, we let $M_\lambda := \bigcup_{\gamma<\lambda} M_\gamma$, and we let the plural domain be $\bigcup_{\gamma<\lambda} \wp^+(M_\gamma)$. Notice that there is no universal plurality at any limit stage: every plurality at a limit stage is available at, and thus bounded by, some stage below. Predicates are interpreted by taking the union of their interpretations in the preceding models. Likewise, the operator § is interpreted by taking the union of its interpretations in preceding models. Notice that this yields an ordinary, complete $\mathcal{L}^+$-model. Finally, it is routine to verify that at every limit stage, the universal closures of (AP) and (Inher) are true.

All that remains is to ensure that the axioms of Critical Plural Logic are satisfied. To this end, we iterate the construction through all the set-theoretic ordinals. This means that our ultimate limit model, $\mathcal{M}_\infty$, defined as the union of all the $\mathcal{M}_\gamma$, will be a proper class.[63] It is not hard to verify that $\mathcal{M}_\infty$ satisfies all the axioms of Critical Plural Logic. This is straightforward for the principles of Singletons, Adjunction, and Infinity. For Generalized Union, consider a set x of "tags", each of which is applied to the elements of some set. We need to show that the union of all these "tagged" objects is in the plural domain of $\mathcal{M}_\infty$. This follows because, by the set-theoretic axiom of Replacement, there is an ordinal γ such that all the tagged objects are in M_γ, which means that the desired union set is in $\wp^+(M_\gamma)$. $\dashv$

Proposition 5 can be extended in various ways. First, it is straightforward to handle predicates other than φ^*. More interestingly, we can allow any number of different forms of abstraction to proceed in parallel (say, set and cardinality abstraction, side by side). To do so, we modify the above proof by using multiple copies of $\wp^+(M_\gamma)$ when defining $\mathcal{M}_{\gamma+1}$ in terms of $\mathcal{M}_\gamma$. We also require principles governing mixed identities, $\S_1 xx = \S_2 yy$, for distinct abstraction operators $\S_1$ and $\S_2$. As before, these principles must enable us to resolve such identities based only on information concerning the previous model $\mathcal{M}_\gamma$. A particularly simple and natural option is to deem all mixed identities false (say, that no set is identical with a cardinal number).

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[63] This proper class can be handled as in [Kunen, 1980]. Alternatively, we could assume the existence of a strongly inaccessible cardinal κ and show that the set-theoretic model $\mathcal{M}_\kappa$ satisfies the axioms of Critical Plural Logic.

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