1. LAND ACKNOWLEDGEMENT

This seminar takes place on and the work I have done preparing my talk for this seminar occurred on traditionally Abenaki land.

2. ABSTRACT

In this talk, I will introduce some tools used to compute the canonical ring of a smooth, projective variety. First, I define line bundles and their induced maps to projective space and then discuss geometric conditions for a line bundle to be very ample and therefore to define an embedding. I also give a classic and a derived functor definition of the canonical bundle with the intention of explicitly using the adjunction formula to compute the genus of complete intersections. There will be a motivating example throughout the discussion: realizing the complete intersection of three quadrics as a genus five curve, and the converse embedding of a genus five curve as a complete intersection of three explicit quadrics.

3. INTRODUCTION

What is embedding about?

\[
\begin{pmatrix}
\text{circles} \\
\text{as abstract things}
\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}
\text{equations,} \\
\text{trig,} \\
\text{ect \ldots}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{varieties} \\
\text{e.g. curves} \\
\text{e.g. surfaces}
\end{pmatrix} \xrightarrow{\sim} \begin{pmatrix}
\text{models} \\
\text{rings} \\
\text{syzygies}
\end{pmatrix}
\]

The main technique for embedding is to find a big enough space to fit your variety in to, but one that is as small as possible while accomplishing that goal. In particular the target space for embeddings in this talk is projective space, since as a Riemann surface projective space is compact, but also has coordinates, so we have variables to write down our equations, and we want to use as few variables as possible in our models.

The technique to accomplish an embedding is to consider a line bundle on the variety.

Etymology:
bundle: A line bundle is an invertible sheaf, from which the bundle part of the phrase arises, and in practice the thing which matters about being a sheaf is the transition data between open sets.

line: The line part of the line bundle specifies that in particular we consider an (invertible) sheaf of rank 1 modules over some ring. The computational tool which makes working with line bundles particularly convenient is the equivalence between invertible sheaves and effective divisors.

**Definition 3.1.** A **divisor** on some variety \( X \) is a formal sum of codimension 1 subspaces of \( X \).

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**4. The Embedding Associated to a Line Bundle**

How does a line bundle work?

\[
\begin{align*}
\mathcal{L} & \rightarrow X \\
l & \mapsto p \\
some \text{line} & \mapsto \text{a point}
\end{align*}
\]

The lines which lie over a neighborhood of a point of the variety correspond under the sheaf of rank 1 modules to some rational sections, or some polynomials which cut out submodules. Those sections which are defined on the particular open set \( |X| \) itself are known as global sections. Given a line bundle \( \mathcal{L} \) with some global sections \( \langle s_0, \cdots, s_N \rangle = H^0(X, \mathcal{L}) \) for some \( N \) there is an associated map to projective space

\[
\varphi_{\mathcal{L}} : X \rightarrow \mathbb{P}^{N-1} \\
p \mapsto (s_0(p), \cdots, s_N(p))
\]

and this idea completely makes sense since a line bundle is by definition an invertible sheaf, so whatever maps (lines) \( \sim \) (points) we had from the line bundle, we should have inverse maps the other direction, and for a geometer a line is \( \mathbb{P}^1 \) the projective line in particular. Note for experts everything in this talk is smooth and hence all the projective stuff.

Note: we embed into projective space since as a complex manifold this is a compact space and we can extend theorems about solutions to polynomials, computation of derivatives and evaluations of integrals to this space thoroughly.

**5. Well-definedness/When does the line bundle actually do what we want?**

**Definition 5.1.** Say that a line bundle \( \mathcal{L} \rightarrow X \), for \( X \) some smooth projective variety of dimension \( n \), is **very ample** if

1. **the induced map** \( \varphi_{\mathcal{L}} : |X| \rightarrow \mathbb{P}^n \) **of underlying topological spaces is a closed immersion**
   (note: thinking of \( \mathcal{L} \sim D \) for some effective \( D \in \text{Div}(X) \) this is the condition that for all \( x, y \in X \), \( h^0(D - x - y) < h^0(D - x) < h^0(D) \))
2. \( \varphi_{\mathcal{L}} \) ‘separates points’ so the map on the spaces is injective
   (note: now for all \( x \neq y \in X \), \( h^0(D - x - y) < h^0(D - x) < h^0(D) \))
3. \( \varphi_{\mathcal{L}} \) ‘separates tangent directions’
   (note: this happens when \( h^0(D - 2x) < h^0(D - x) < h^0(D) \) for each point \( x \in X \))
notes:

(1) the last condition is similar to an inverse function theorem statement about invertibility of the map
(2) this also avoids contracting some tangency information to an infinitesimal neighborhood in the image which might have some nilpotents
(3) closed immersion $\Rightarrow$ existence of sections.

**Definition 5.2.** A map $X \xrightarrow{f} Y$ is a closed immersion if there exists an affine open cover $Y = \cup_{i=1}^{n} U_i$, $U_i = \text{Spec}(R_i)$ such that for each $i$ there exists some ideal $I_i \subset R_i$ with $f^{-1}(U_i) = \text{Spec}(R_i/I_i)$ as schemes over $U_i$.

In particular this means $f$ induces a map $\|X\| \xrightarrow{\sim} \left( \text{closed subset of } Y \right)$ with $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ surjective and $\ker(f^\#) \subset \mathcal{O}_Y$ a quasi-coherent sheaf of ideals locally generated by sections.

Another definition of very ample from stacks project is

**Definition 5.3.** Say a line bundle $L$ is very ample if the embedding $\varphi_L : X \rightarrow \mathbb{P}^r$ by global sections of $L$ is a closed immersion and $L$ is basepoint free.

6. Picking a good line bundle

The particular line bundle we will consider is the canonical bundle $\omega_X$. One definition of this is the “top exterior power of the sheaf of differentials” written $\omega_X = \Omega^n_X = \bigwedge^n \Omega^1_X$, when $\dim(X) = n$.

Another more general and abstract definition for $X \subset \mathbb{P}^N$ a projective, connected and purely $n$-dim variety over some algebraically closed field $k$ is

$$\omega_X = \text{Ext}^{N-n}_{\mathcal{O}_\mathbb{P}^N}(\mathcal{O}_X, \omega_{\mathbb{P}^N}).$$

In this treatment $\omega_X$ is really thought of as a dualizing sheaf. This is the right idea since we have first of all that $T_X^\vee \cong \Omega^1_X$ for $T_X$ the tangent bundle, and for experts this lines up nicely with the ideal sequence if we take Serre duals everywhere, but also gives us a way to compute canonical bundles by invoking Serre duality on global sections for subschemes (i.e. this sets up adjunction).

Note: $\omega_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}(-N-1)$ which we will probably use with adjunction later but this is a good example of a twisted line bundle.

7. Adjunction

i.e. actually writing down some canonical bundles

**Theorem 7.1.** [2 29.3.1](Adjunction formula)

Let $D$ be a smooth divisor on a smooth $X$ over some field $k$. Then $\omega_D = \omega_X(D)|_D$.

(Note: normal bundle in the sense of vector bundle of normal vectors to the variety)

As a corollary, if $Y \subset X$ is a smooth subscheme with normal bundle $N_Y$ then

$$\omega_Y \cong \omega_X|_Y \otimes \det(N_Y),$$

and this particular form gives us a nice way to describe complete intersections.
8. Derived functor/Dualizing sheaf

Running somewhat rough-shod over a few intermediate definitions still motivates why the derived functor/dualizing sheaf formulation of the canonical bundle is the better definition. Informally then, $X$ is Cohen-Macauay if there is some regular sequence which cuts $X$ down to 0-dim. Then if $X$ is Cohen-Macaulay and $\omega_X \cong \mathcal{O}_X$ locally outside of singularities (i.e. the canonical divisor is Cartier) then say that $X$ is Gorenstein.

**Example.** Say $X \subset \mathbb{C}^3$ is some hypersurface with an isolated singularity at $P$. If we sketch $X$ some singular normal surface

![Diagram](image)

then

$$
\begin{align*}
\omega_X &= \Omega^1_X \quad \text{maybe torsion} \\
&= \bigwedge^2 \Omega^1_X \quad \text{still maybe torsion} \\
&= (\bigwedge^2 \Omega^1_X)^\vee \quad \text{-taking reflexive hull} \\
&= \mathcal{O}_X(K_X) \quad \text{locally free}
\end{align*}
$$

Expressing this in a manner which is supposed to evoke adjunction we say

$$
\omega_X = \omega^3_{\mathbb{C}^3}(X)|_X
$$

(normally we’d say $K_X = (K_{\mathbb{C}^3} + X)|_X$) we have this well-defined notion of a line bundle on $X$ and outside of $\text{Sing } X$ this is just $\Omega^2_X$. Finally if we allow analysis to rear it’s ugly head and take some Poincare residues (Jesse: [see [7] for Poincare polynomials; this is a weighted projective space idea]) and do some implicit function theorem, then the $\text{Res}_X f$ is a rational 2-form on $X$ which is a basis of $\Omega^2_X|_{X-\text{Sing } X}$ and $\Omega^2_X \cong \mathcal{O}_X$ outside of $P$. Then we just define $\omega_X$ to be $\mathcal{O}_X$ extended (analytically continued if you insist) over $P$ with a basis $\text{Res}_X f$.

The reason for setting up the derived functor $\text{Ext}$ definition of the canonical bundle is to get an automatic version of this adjunction property, i.e. to think of a dualizing sheaf. We define the derived functor as follows:

Suppose $X \subset Y$ is some codim 1 subspace. Consider the usual ideal sequence of coherent sheaves on $Y$

$$
0 \to \mathcal{I}_X \to \mathcal{O}_Y \to \mathcal{O}_X \to 0
$$

and take $\text{Hom}_{\mathcal{O}_Y}(-, \omega_Y)$’s on the sequence (remember $\text{Hom}$ is contravariant).

$$
0 \to \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y) \to \text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y) \to \text{Hom}_{\mathcal{O}_Y}(\mathcal{I}_X, \omega_Y) \to \text{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y) \to \text{Ext}^1_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y) \to \cdots
$$

and we make the following observations:

1. $\text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, \omega_Y) = 0$ since $\mathcal{O}_X$ is supported on $X$ so as far as $Y$ is concerned any $s \in \mathcal{O}_X$ is torsion but $\omega_Y$ is torsion free by definition
2. $\text{Hom}_{\mathcal{O}_Y}(\mathcal{O}_Y, \omega_Y) = \omega_Y$ since $\mathcal{O}_Y$ is projective so is killed by derived functors
Thinking about (2) more the condition about being killed by derived functors comes from

\[ 0 \to \omega_Y \to \omega_Y(X) \to \omega_X \to 0, \]

where \( \omega_Y(X) = \text{Hom}(\mathcal{I}_X, \omega_Y) \) so if \( Y \) is generically nonsingular

\[ 0 \to \Omega_Y^{(n+1)}(X) \to \Omega_Y^{(n+1)}(X) \to \Omega_X^n \to 0. \]

♠♠♠ Jesse: [these sequences are automatic adjunction]

The upshot here is that \( \omega_X \) and the usual manipulations one does to it have the nice adjunction properties of \( \mathcal{O}_X(K_X) = \Omega_X^n \) but in particular the derived functor statement extends to singular surfaces as well.

9. Back to adjunction: the punchline calculation

In particular from Exercise [2, 28.1.3] if \( Y = D_1 \cap \cdots \cap D_r \subset \mathbb{P}^n \) is some complete intersection then

\[ N_Y = \mathcal{O}_{\mathbb{P}^n}(D_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(D_r). \]

From this Taylor arrives at a genus formula for complete intersections in Example [2, 29.4.6]: if \( D_1, D_2, D_3 \subset \mathbb{P}^4 \) are hypersurfaces of degrees \( d_1, d_2, d_3 \) respectively and \( Y \) is the complete intersection of the \( D_i \)'s then

\[ g_Y = \frac{(d_1 + d_2 + d_3 - 5)d_1d_2d_3 + 2}{2}. \]

Therefore the complete intersection of three quadrics in \( \mathbb{P}^4 \) is a genus 5 curve. Note that this statement actually works both ways, and a canonical genus 5 curve embeds in \( \mathbb{P}^4 \) as the complete intersection of 3 quadrics. That proof is somehow more laborious, involving basepoint free pencils, explicit computation of top exterior powers of the tangent bundle (i.e. explicit Koszul cohomology), Noether’s theorem about projective normality, and involved vanishing theorems in sheaf cohomology. A full treatment is given at [https://www.uvm.edu/~jfrankl2/](https://www.uvm.edu/~jfrankl2/) in a document called “Thesis proposal about Petri’s theorem with open problems” but a summary is mentioned here.

END of talk

10. The converse statement that genus 5 curves are the intersection of 3 quadrics in \( \mathbb{P}^4 \) and all that

Proof. (sketch)

The specific computation in the document mentioned is more algebraic than geometric in a lot of ways, and has to do with the canonical ring \( R(X) := \bigoplus_{d=0}^\infty H^0(X, \omega_X^\otimes d) \), where \( X \) is a genus 5 curve. From some other theory, \( R \) is finitely generated and has form \( R \cong \mathbb{C}[x_0, \cdots, x_4]/I \) where \( I \) is itself a finitely generated, graded ideal of \( R \). The 3 quadrics whose complete intersection is the image of \( X \) under the canonical embedding \( \varphi_{\omega_X} \) are called syzygies, which in particular form a basis for \( I_2 \) the homogeneous degree 2 component of the canonical ideal \( I \). The main point of the proof is to argue that because of the vanishing theorems in sheaf cohomology and the Koszul cohomology sequences which intersect those truncated long exact sequences transversally, the snake lemma proves that \( I_2 \) surjects onto \( I_3 \) and the entire canonical ideal is generated by the 3 degree 2 homogeneous equations which we compute in forming a basis \( R \).

What is really happening behind the scenes in this computation comes down to a few details which can be introduced without too much overhead:
(1) for \( D \in \text{Div}(X) \) of form \( D = p_1 + \cdots + p_5 \) a sum of points in general position, a basis \( s_1, \ldots, s_5 \) of \( H^0(X, \omega_X) \) can be defined by the conditions
\[
\begin{cases}
s_i(p_i) \neq 0 \\
s_i(x_j) = 0, & \text{if } i \neq j.
\end{cases}
\]
Note that verifying this basis is well-defined is a heinous triple subscript type of computation written in [3, 1.6]. I think of these elementary vanishing conditions on global sections as generating \( I_1 \) the first graded piece of the canonical ideal.

(2) \( I_2 \)'s basis comes from explicitly writing down all the possible products of the sections \( s_i \) from before. It can be shown (see the treatment after [3, 1.2]) that only 3 different degree 2 homogeneous equations of the form \( s_is_j \) are needed to generate the rest of this component of the ideal. The basepoint free pencil trick in [1] comes up.

(3) Those relations form a finitely generated graded ring with its own set of generators and relations, and it can be shown inductively that the new relations form the same sort of structure, but everything comes down to the elementary relations in \( I_2 \) derived in the previous step.

(4) Some rather crude feeling geometric considerations align this ring-theoretic description with Koszul cohomology and the some twisted version of the Euler ideal sequence to set up a snake lemma proof that \( I_2 \) generates \( I \).

(5) Note that a key lemma in the previous step is the normality of the embedded curve, which is handled separately and involves itself several vanishing lemmata in sheaf cohomology. This the business found in the Noether's theorem section.

\[ \square \]

11. Basepoint free pencils

First, let me state a paraphrase of Arbarello et al.'s version of the basepoint free pencil trick. Informally, a pencil is some compatible collection of fibers of maps to \( \mathbb{P}^1 \). In some open subset of \( \mathbb{P}^1 \), above each point there is some kind of fiber (either a point or curve) which lies on the curve or surface mapping to \( \mathbb{P}^1 \) with a common condition such as contracting to the point on \( \mathbb{P}^1 \) or vanishing there to a specific order.

**Theorem 11.1.** [1] **Basepoint free pencil trick** Let \( C \) be a smooth curve, let \( L \) be an invertible sheaf on \( C \). Suppose \( s_1 \) and \( s_2 \) are linearly independent sections of \( L \) and denote the subspace of \( H^0(C, L) \) which they generate \( V \). Then the map
\[
\phi_{2,2} : V \otimes H^0(C, \mathcal{O}_C) \rightarrow H^0(C, \mathcal{O}_C \otimes L)
\]
given by
\[
s_1 \otimes t_2 - s_2 \otimes t_1 \mapsto s_1t_2 - s_2t_1
\]
has kernel
\[
\ker \phi_{2,2} \cong H^0(C, \mathcal{O}_C \otimes L^{-1}(B)),
\]
where \( B \) is the base locus of the pencil spanned by \( s_1 \) and \( s_2 \).

The basepoint free pencil trick on curves is a baby version of the full tool, where independence of sections which locally generate the line bundle everywhere is controlled by some prescribed orders of vanishing of those sections at a mobile linear system of points on the curve. For surfaces, the point is somehow to show that there is no pencil of curves prescribed by some common singular point on the surface which lies in a mobile linear system over a \( \mathbb{P}^1 \) in the target. Since the surface case has a neater sketch, we turn towards that version next.

**Kollar's Slogan:** vanishing is when a coherent cohomology group is controlled by a topological cohomology group.
Consider maps res\(_D : H^1(O_S) \to H^1(O_D)\) on \(S\) some nonsingular, minimal projective surfaces over \(\mathbb{C}\) of general type as usual. Notably, ker(res\(_D\)) doesn’t really depend on \(D\) in the following sense. If \(\hat{C} \to D_{\text{red}}\) is the nonsingular model of the reduced curve \(D_{\text{red}}\) (i.e. the resolution of singularities which is a blowup at finitely many points), then In pictures, this reduction of \(D\) and

\[
\begin{array}{c}
H^1(O_S) \xrightarrow{\text{res}} H^1(O_D) \\
\downarrow \\
H^1(O_S) \xrightarrow{\text{res}_{D_{\text{red}}}} H^1(O_{D_{\text{red}}}) \\
\downarrow \\
H^1(O_{\hat{C}}).
\end{array}
\]

then resolution of singularities looks something like the following diagram.

Theorem 11.2. *(Ramanujan vanishing)*
If \(S\) is defined over a field of characteristic 0 then ker(res\(_D\)) = ker(res\(_{D_{\text{red}}}\)) = ker(res\(_{\hat{C}}\)).

This is a subtle proof and I’m not going to touch it but rather just black box this guy and go back to the discussion about singular points not being included in the base locus on surfaces. Assume that \(D''\) the divisor defined by \(D'' = f^*D - 2E\) on the blowup \(S_1\) is also big so \((D'')^2 > 0\). Then for sufficiently large \(N\), \(ND''\) moves in a nontrivial and large linear system. For \(H\) some very ample divisor \(H^1(ND'' - H) \neq 0\) or in other words \(ND'' - H\) is effective and we get maps

\[
H^1(O_S(-H)) \to H^1(O_S) \xrightarrow{\text{res}} H^1(O_H)
\]

and \(H^1(O_S) \to H^1(O_{ND''})\) where Serre vanishing (see why we needed \(N >> 0\)?) gives \(H^1(O_S(-H)) = 0\). We conclude that

\[
H^1(O_S) \to H^1(O_{ND''})
\]

which demonstrates the earlier claim that if \(D\) is numerically 2-connected and \(D^2 > 4\), then for \(P \in \text{Sing}(D)\), \(P \notin \text{Bs}\ |K_S + D|\).

Theorem 11.3. *(Francia vanishing)*
Let \(S\) be a nonsingular projective surface over a field of characteristic 0. Suppose \(D\) is an effective divisor on \(S\) and \(P \in \text{Sing}(D)\). Then \(P \notin \text{Bs}\ |K_S + D|\) if and only if \(H^0(O_D) \to H^0(O_{D''})\).

Example. \(D = C_1 + C_2\) with \(C_1C_2 = 1\), sketched below.
Here \(P \in \text{Bs}\ |K_S + D|\) since \((K_S + D)|_{C_1} = K_S + C_1 + C_2|_{C_1}\) by adjunction. But if \(C_1\) is
nonsingular then when we consider the Riemann Roch spaces $\mathcal{L}(K_{C_1} + P) = \mathcal{L}(K_{C_1})$ by the Cauchy residue formula, since the integral around a nonsingular point vanishes.

**Criterion for a very ample linear system/line bundle**

$X$ projective scheme

$H$ line bundle (Cartier divisor)

pick a basis $H^0(\mathcal{O}_X(H)) \ni x_1, \ldots, x_n$ and the game is to consider the following map

$$\varphi_H : X \to \mathbb{P}^n$$

mapping $p \mapsto (x_i(p))$.

It is well known $X \cong \overline{X} \subset \mathbb{P}^n$ if and only if

1. for each $p \in X$ a closed point, there is some $x_i \notin H^0(m_p \cdot \mathcal{O}_X(H))$ where nevertheless $x_i \in H^0(\mathcal{O}_X(H))$, i.e. $x_i$ is a section which is locally a basis

$$\mathcal{O}_X(H) \quad \text{locally free}$$

$$x_i \in \mathcal{O}_X(H)$$

2. for each $P \neq Q \in X$ there exists two linear combinations $\zeta_1(x), \zeta_2(x)$ such that $\zeta_1(P) \neq 0, \zeta_1(Q) = 0; \zeta_2(P) = 0, \zeta_2(Q) \neq 0$,

or in common parlance “$\varphi_H$ separates points”

In this case taking the sheaf cohomology of the exact sequence

$$0 \to m_P \cdot m_Q \cdot \mathcal{O}_X(H) \to \mathcal{O}_X \to \kappa_P + \kappa_Q \to 0$$

there’s a surjection $H^0(\mathcal{O}_X) \to H^0(\kappa_P + \kappa_Q)$ mapping $\zeta_1$ and $\zeta_2$ to a basis (say $(1,0)$ and $(0,1)$).

3. the “separates tangent directions” condition considers $\mathcal{I} \subset m_P \subset \mathcal{O}_X$ some ideal of colength 1. Geometrically, $\mathcal{I}$ corresponds to some codimension 1 subspace of $m_P/m_P^2$ which is of course dual to the tangent space $T_{X,P}$ at $P$. Here the condition is that there is some $x_1(P) \neq 0, x_2(P) = 0$ and $x_2 \notin \mathcal{I} \cdot \mathcal{O}_X(P)$. In this case

$$0 \to \mathcal{I} \cdot \mathcal{O}_X(H) \to \mathcal{O}_X(H) \to k[t]/(t^2 = 0) \to 0$$

being exact is the condition we need.

Everyone knows how to prove this, we did it in Taylor’s class, just use coherence of sheaf cohomology, Nakayama, algebra and whatnot, and I’m not ready to talk about basepoints on curves right now, there’s good surface stuff to do first.

$S$ nonsingular, projective, minimal surface of general type over $\mathbb{C}$.

(recall: gen type means $K_S$ is nef and $K^2 > 0$)

Consider the linear system $|K_S|$ of dimension $p_g - 1$ and pick some basis $H^0(K_S) \ni x_1, \ldots, x_{p_g}$. 
Definition 11.4. The base locus is the ideal $\mathcal{I} \subset O_S$ with a map $H^0(K_S) \to H^0(\mathcal{I} \cdot O_S(K_S))$ such that for every point there’s some $x_i$ a local basis. The ideal $\mathcal{I}$ can vanish on points or curves on $S$ and

$\mathcal{I} \otimes O_S(K_S) = \left( \begin{array}{c} \text{image of} \\ O_S \otimes H^0(K_S) \end{array} \right)$.

If $|K_S| = F + M$ for $F$ some fixed part and $M$ some mobile part which moves without fixed components, $\mathcal{I} \subset O_S(-F) \subset O_S$.

We’ll consider the case when $|K_S| = F + M$ for $F$ and $M$ as above, and in particular when $|K_S|$ has some base curves. Think as well of $p_g$ comparatively large with respect to $K^2$ like a Horikawa surface with $p_g = 3$, $K^2 = 2$. Using intersection pairing compute

$K^2 = KF + KM \geq KM$, since $K \cdot F \geq 0$ by nef

and

$KM = (F + M)M \geq M^2$, since $FM \geq 0$.

The mobile linear system $M$ on $S$ defines a map $\varphi_M : S \to \mathbb{P}^{p_g-1}$ whose image $\varphi_M(S)$ is some subvariety which spans $\mathbb{P}^{p_g-1}$, i.e. is not contained in any hyperplane.

We want $\varphi^*H^0(O_{\mathbb{P}}(1)) \cong H^0(K_S)$ but we’ll consider in particular a pencil of curves which messes this up. Suppose $\varphi_M(S)$ is some curve. Then a general element of $M$ the mobile linear system has $(p_g - 1)$ irreducible components and we say $|K_S|$ is composed with a pencil. The sketch is below.

Addition topics and definitions that might be important

1. line bundle = invertible sheaf equiv to effective divisor:

   $\text{div}(X) \ni D \rightarrow \text{O}_X(D)$

   $(\mathcal{L}, s) \rightarrow \text{div}(s)$

   note: this is really a map $\mathcal{L} \rightarrow c_1(\mathcal{L})$ the first Chern class, $s$ some rational section

2. TBD
Comp Exam
♠♠♠ Jesse: [Taylor’s idea is to finish the complete intersection of 3 quadrics in \( \mathbb{P}^4 \) with adjunction into genus formula so I’m writing that up]

Title for Grad Student Seminar
Not your grandma’s canonical embedding

Abstract
Let \( X \) be a genus \( g \geq 3 \), smooth, irreducible, projective, algebraic, canonical curve over \( \mathbb{C} \) and denote the canonical bundle on \( X \) by \( \omega = \omega_X \). The canonical ring of \( X \) is the ring

\[
R = R(X, \omega) = \bigoplus_{n \geq 0} H^0(X, \omega^\otimes n) \cong \mathbb{C}[x_1, \cdots, x_g]/I,
\]

where \( I \) is a finitely generated ideal generated in bounded degrees. Typically, canonical embedding is about describing the nature of the canonical ideal \( I \) for a given abstract variety \( X \), but in this talk we instead begin with a complete intersection \( X \subset \mathbb{P}^4 \) and calculate the unique numerical invariant for the curve \( X \), its genus.

12. Background

Let \( X \) be some canonical variety over \( \mathbb{C} \) (resp. algebraically closed and characteristic 0 to minimize pfaffing) and denote the canonical bundle on \( X \) by \( \omega = \omega_X \). Note that if \( \dim(X) = n \) and we conflate \( X \) with its image under the canonical embedding \( \varphi_\omega : X \to \mathbb{P}^N \), then we can express

\[
\omega = \bigwedge^n \Omega^1_X = \text{Ext}^{N-n}_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}),
\]

where \( \Omega^1_X \) is the sheaf of regular differential one forms on \( X \), and \( \text{Ext} \) is the derived functor of sheaf \( \text{Hom}_{\mathcal{O}_{\mathbb{P}^N}}(\cdot, \omega_{\mathbb{P}^N}) \). In the first presentation we’re thinking of the classical top exterior power of the sheaf of differentials, which is canonical in the sense of the convention that tensor products of more than \( n \) terms of a rank \( n \) module vanish, so we’re taking as big of an exterior power as possible. In the latter presentation, we view \( \omega \) as the dualizing sheaf (we literally took Hom’s) and this version gives us an automatic adjunction property from the ideal sequence for \( X \)

\[
0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0.
\]

13. Statement of the problem

What do we want to compute?

**Definition 13.1.** [7] The canonical ring of \( X \) is the ring

\[
R = R(X, \omega) = \bigoplus_{n \geq 0} H^0(X, \omega^\otimes n).
\]

**Fact 13.2.** It turns out that not only is \( R \) finitely generated, but we can even express

\[
R \cong \mathbb{C}[x_1, \cdots, x_r]/I,
\]

for \( r = g = \text{genus}(X) \) when \( X \) is a curve (even stacky) or some other finite \( r \) in the case of surfaces, and \( I \) some ideal which in particular has the form of some \( \mathbb{C} \)-algebra (i.e. is a graded ring in its own right). Further, we even know that \( I \) is finitely generated, by a minimal generating set in bounded degrees.

We want to describe \( I \) as clearly as possible, or respectively, starting from a description of \( I \), to determine what \( X \) is as clearly as possible.
**Example.** Our motivating example will be the variety $X \subseteq \mathbb{P}^4$ which is the complete intersection of 3 quadrics $D_1, D_2, D_3$. By complete intersection I mean the following:

$$X = D_1 \cap D_2 \cap D_3,$$

in particular in such a way that if $D_i = D_{+}(f_i)$ for $f_i$ a degree $d_i = 2$ polynomial in 5 variables (from $\mathbb{P}^4$) then

$$I = \langle f_1, f_2, f_3 \rangle.$$

14. Adjunction

i.e. actually writing down some canonical bundles.

In this section we will use a powerful computational tool called adjunction to compute the canonical bundle on a subscheme of $\mathbb{P}^N$ for a nice example. The aim is to describe the canonical bundle $\omega$ on our variety $X$ by conflating $X$ with $\varphi_\omega(X) \subseteq \mathbb{P}^N$ and using adjunction on the embedded subscheme.


**Theorem 14.1.** [2] 29.3.1 (Adjunction formula)

Let $D$ be a smooth divisor on a smooth $X$ over some field $k$. Then $\omega_D = \omega_X(D)|_D$.

(Not: normal bundle in the sense of vector bundle of normal vectors to the variety)

**Corollary 14.2.** [2] If $X \subseteq \mathbb{P}^N$ is a smooth subscheme with normal bundle $N_X$ then

$$\omega_X \cong \omega_{\mathbb{P}^N}|_X \otimes \det(N_X).$$

This latter form gives us a nice way to describe complete intersections because we know a lot about $\mathbb{P}^N$.

**Definition 14.3.** [2] 29.1 For $E$ a locally free sheaf of rank $r$, define $\det(E) = \bigwedge^r E$.

**Theorem 14.4.** [2] 29.1.1 Let $E, F, G$ be locally free sheaves on $Y$ and suppose $f : X \to Y$ is a morphism of schemes. Then if $0 \to E \to F \to G \to 0$ is exact then

$$\det(E) \otimes \det(G) \cong \det(F).$$

**Proof.** This is Hartshorne II.6.11. \[Q.E.D.\]

In particular for $X = D_1 \cap D_2 \cap D_3 \subseteq \mathbb{P}^4$ a complete intersection of degree $\deg D_i = d_i$ hypersurfaces and $f = \varphi_{K_X} : X \to \mathbb{P}^4$ the usual canonical embedding, we consider the exact sequence of line bundles

$$0 \to \mathcal{O}_{\mathbb{P}^4}(D_1)|_X \to \mathcal{O}_{\mathbb{P}^4}(D_1)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_2)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_3)|_X \to \mathcal{O}_{\mathbb{P}^4}(D_2)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_3)|_X \to 0$$

where $i$ is the inclusion map $i = (id \mathcal{O}_{\mathbb{P}^4}(D_1)|_X, 0, 0)$ and $pr$ is projection onto the second and third coordinates and deduce

$$\det(\mathcal{O}_{\mathbb{P}^4}(D_1)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_2)|_X \oplus \mathcal{O}_{\mathbb{P}^4}(D_3)|_X) \cong \det(\mathcal{O}_{\mathbb{P}^4}(D_1)|_X) \otimes \det(\mathcal{O}_{\mathbb{P}^4}(D_2) \oplus \mathcal{O}_{\mathbb{P}^4}(D_3)|_X)
= \mathcal{O}_{\mathbb{P}^4}(d_1)|_X \otimes \mathcal{O}_{\mathbb{P}^4}(d_2)|_X \otimes \mathcal{O}_{\mathbb{P}^4}(d_3)|_X
= \mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3)|_X.$$
14.2. Projective space in particular.

**Fact 14.6.** Some useful things we know about $\mathbb{P}^N$:

1. $\omega_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}(-N - 1)$.
2. $N_X = \mathcal{O}_{\mathbb{P}^n}(D_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^n}(D_r)$ (in the complete intersection situation we’re using as an example)
3. $\mathbb{P}^N$ can be covered with exactly two charts

**Proof.** See [3] for a proof of part 1 in the case $N = 2$. I’ll attempt to generalize that. Note that the proof of part 3 shows up when we use the fact that there are always unique sections in the compliment of a given chart as we consider a typical cover of $\mathbb{P}^N$ by $N + 1$ charts corresponding to the vanishing of distinct projective coordinates.

Say we have charts $U_i = \{x_i \neq 0\}$ for $i = 0, \cdots, N$ on $\mathbb{P}^N = \text{Proj} k[x_0, \cdots, x_N]$. In particular on $U_i$ there are coordinates $u_0, \cdots, u_{i-1}, u_{i+1}, \cdots, u_N$ where

$$(x_0; x_1; \cdots; x_N) = (u_0; \cdots; u_{i-1}; 1; u_{i+1}; \cdots; u_N) \Rightarrow u_j = \frac{x_j}{x_i}, \ i \neq j.$$

Using Lemma 14.5 locally on some $U_i$ the sections of $\omega_{\mathbb{P}^N}$ have form $f(u_0, \cdots, \tilde{u}_i, \cdots, u_N)du_0 \wedge \cdots \wedge \tilde{du}_i \wedge \cdots \wedge du_N$ and since there is only one section away from from $U_i$, namely $x_i = 0$, it suffices to check zeros and poles of $f$ there. On $U_j$ for some $j \neq i$ we have the transition data

$$(u_0; \cdots; u_{i-1}; 1; u_{i+1}; \cdots; u_N) = (u_0; \cdots; u_{j-1}; 1; u_{j+1}; \cdots; u_N)$$

so for convenience change the latter coordinates to $v$’s so it is easier to see how we make the next computation. Then we have

$$u_k = \begin{cases} \frac{u_k}{v_k}, & k \neq j \\ \frac{1}{v_k}, & k = j \end{cases}$$

and compute on $U_j$ that

$$du_0 \wedge \cdots \wedge \tilde{du}_i \wedge \cdots \wedge du_N = \frac{(-1)^N}{v_j^{N+1}}dv_0 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_N$$

which has a pole of order $N + 1$ when $v_j = 0$. \hfill \Box

14.3. **Computation: the genus formula.** Now we invoke the heavy hammer known as the Riemann-Roch theorem, together with the facts above and the adjunction formula to back-calculate $X$ from its description as a complete intersection of 3 quadrics in $\mathbb{P}^4$.

Let $K_X \in \text{Div}(X)$ denote the canonical divisor (class) of $X$ as in the motivating example. Then by Riemann-Roch

$$\deg(K_X) = 2g - 2.$$

We’ll use adjunction to determine the canonical divisor $K_X$ we need to use Riemann-Roch. We’ll do this more generally for $X$ the complete intersection of 3 degree $d_i$ hypersurfaces $D_i \subset \mathbb{P}^4$. One last fact.

**Fact 14.7.** The degree of $X = D_1 \cap \cdots \cap D_r$ some complete intersection of degree $d_1, \cdots, d_r$ hypersurfaces in $\mathbb{P}^N$ is $\deg X = \prod_{i=1}^r d_i$. 
By adjunction we have
\[
\omega_X = \omega_{\mathbb{P}^4} \mid_X \otimes \det(N_X) \\
= \mathcal{O}_{\mathbb{P}^4}(-5) \mid_X \otimes \det(O_X(d_1) \oplus O_X(d_2) \oplus O_X(d_3)) \\
= \mathcal{O}_{\mathbb{P}^4}(-5) \mid_X \otimes \det(O_{\mathbb{P}^4}(d_1) \mid_X \oplus O_{\mathbb{P}^4}(d_2) \mid_X \oplus O_{\mathbb{P}^4}(d_3) \mid_X) \\
= \mathcal{O}_{\mathbb{P}^4}(-5) \mid_X \otimes \mathcal{O}_{\mathbb{P}^4}(d_1) \mid_X \otimes \mathcal{O}_{\mathbb{P}^4}(d_2) \mid_X \otimes \mathcal{O}_{\mathbb{P}^4}(d_3) \mid_X \\
= (\mathcal{O}_{\mathbb{P}^4}(-5) \otimes \mathcal{O}_{\mathbb{P}^4}(d_1) \otimes \mathcal{O}_{\mathbb{P}^4}(d_2) \otimes \mathcal{O}_{\mathbb{P}^4}(d_3)) \mid_X \\
= \mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5) \mid_X \\
= \mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5)d_1d_2d_3 \\
= \mathcal{O}_{\mathbb{P}^4}(1)(d_1 + d_2 + d_3 - 5)d_1d_2d_3.
\]

Then if \( K_X = [\text{div}(\omega_X)] \) is the Cartier divisor class associated to the bundle we just computed then by Riemann-Roch we find the genus of \( X \)
\[
\begin{align*}
\text{deg}(K_X) &= 2g - 2 \\
\text{deg}(\mathcal{O}_{\mathbb{P}^4}(1)(d_1 + d_2 + d_3 - 5)d_1d_2d_3) &= 2g - 2 \\
(d_1 + d_2 + d_3 - 5)d_1d_2d_3 &= 2g - 2 \\
\Rightarrow g &= \frac{(d_1 + d_2 + d_3 - 5)d_1d_2d_3 + 2}{2}.
\end{align*}
\]

REFERENCES


