Comprehensive Exam

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1. Notation and Preliminaries

In this section, we will define a topology which we use throughout these notes, and state one computational theorem. These facts are found in standard treatments such as [5].

For expert readers, we begin by specifying a topos for our schemes, and for the non-expert, we define a topology that we will use on our schemes.

**Definition 1.1.** [5] A subset \( X \subseteq \mathbb{P}^N \) is algebraic if there exists a set \( T \subseteq \mathbb{C}[x_0, \ldots, x_N] \) of homogeneous polynomials such that \( X = Z(T) = \{ P \in \mathbb{P}^N : f(P) = 0 \text{ for all } f \in T \} \). The Zariski topology on \( \mathbb{P}^N \) is defined by open sets which are the complements of algebraic sets.

**Example.** The Zariski topology on \( \mathbb{P}^N \) has a basis in the open sets of form \( D(f) = \{ P \in \mathbb{P}^N : f(P) \neq 0 \} \), the nonvanishing locus of the function \( f \in \mathbb{C}[x_0, \ldots, x_N] \).

By means of defining as little as possible to get as much done as we can, we state only a few things which appear in the detailed anatomy of a sheaf on a scheme.

**Definition 1.2.** Let \( X \) be a scheme over \( \mathbb{C} \) and let \( F \) be a sheaf on \( X \). Then for an open \( U \subseteq X \), the elements \( s \in F(U) \) are called the sections of \( F \), and in particular are called global sections when \( U = X \). Write \( H^0(X, F) \) for the \( \mathbb{C} \)-vector space of global sections of \( F \), and let \( h^0(X, F) \) be defined as \( \dim_{\mathbb{C}} H^0(X, F) \).

We need a notion of functions on our scheme for the theory which follows, which scheme theoretically means defining a sheaf of rings (of functions).

**Definition 1.3.** [5] Let \( A \) be a ring, let \( p \trianglelefteq A \) be a prime ideal and denote the localization of \( A \) at \( p \) by \( A_p \). Suppose \( X \) is scheme over \( \text{Spec} A \). The structure sheaf \( \mathcal{O}_X \) on \( X \) is the sheaf of rings defined on each open \( U \subseteq X \) to be the ring of functions \( s : U \rightarrow \bigsqcup_{p \in U} A_p \) such that for each \( p \in U \), \( s(p) \in A_p \) and for each \( p \in U \) there is some open neighborhood \( V \) of \( p \) contained in \( U \) and elements \( a, f \in A \) such that for each \( q \in V \), \( f \neq q \) and \( s(q) = a/f \) in \( A_q \).

Finally, turning to the sheaves of modules which appear later in the document, we introduce one last purely sheaf-theoretic idea.

**Definition 1.4.** Let \( X \) be a scheme over \( \mathbb{C} \). For \( E \) a locally free sheaf of rank \( r \) on \( X \), the determinant of \( E \) is \( \det(E) \overset{\text{def}}{=} \wedge^r E \), where \( \wedge^r \) denotes the \( r \)th exterior product, i.e. the \( r \)th graded component of the exterior algebra.

Later we will want to compute the determinant of a vector bundle, i.e. a sheaf such as \( E \) in 1.4 which we can do via the following theorem:

**Theorem 1.5.** [2] Theorem 29.1.1] Let \( X \) be a scheme over \( \mathbb{C} \) and let \( E, F, G \) be locally free sheaves on \( X \). If \( 0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0 \) is exact then

\[ \det(E) \otimes \det(G) \cong \det(F). \]

**Proof.** This is Hartshorne Exercise II.6.11. □
2. Curves and Complete Intersections

The true starting point for the theory which we cover in these notes is the definition of a curve. We specify what kind of curve we consider by in particular the notions of a “projective” scheme, an assumption we make about curves, and “complete intersections” which we discuss in great detail. These choices mean we do not have to make a choice about some kind of ambient space in which our curve could live, but rather let us use the geometry of a well-known scheme to study our subschemes, and that we use the power theory of intersections in algebraic geometry respectively. For a more thorough treatment of complete intersections, and intersection theory in general, consider [3].

Definition 2.1. A curve is an integral, smooth, projective, Noetherian, separated, one-dimensional scheme of finite type over $\text{Spec}(\mathbb{C})$.

In particular, by projective, we mean $X$ is an irreducible algebraic set in $\mathbb{P}^N$ with the induced subset topology. With this notion, we can begin to introduce ring-theoretic objects associated to a curve.

Definition 2.2. [5] Suppose $X$ is any subset of $\mathbb{P}^N$. The homogeneous ideal of $X$, denoted $I(X)$, is the ideal generated by $\{f \in \mathbb{C}[x_0, \ldots, x_N] : f$ is homogeneous and $f(P) = 0$ for all $P \in X\}$.

Next, by using the ideal above, we begin a special case of intersection theory.

Definition 2.3. [5] A variety $X \subset \mathbb{P}^N$ of dimension $n$ is a (strict) complete intersection if $I(X)$ can be generated with $N - n$ elements. We say $X$ is a set-theoretic complete intersection if $X$ can be written as the intersection of $N - n$ hypersurfaces.

The next result allows us to compute the degree of a complete intersection of hypersurfaces of known degrees.

Theorem 2.4. [3, Corollary 1.24] If $c$ hypersurfaces $Z_1, \ldots, Z_c \subset \mathbb{P}^N$ meet in a scheme $X$ of codimension $c$ with irreducible components $C_1, \ldots, C_t$ then

$$\sum \deg[C_i] = \prod \deg[Z_i].$$

Corollary 2.5. The degree of a complete intersection of hypersurfaces $D_1, D_2$ and $D_3 \subset \mathbb{P}^N$ of degrees $d_1, d_2$ and $d_3$ respectively, which intersect in a curve, is $d_1 d_2 d_3$.

Proof. Since a curve is 1-dimensional, it has codimension $N$ in $\mathbb{P}^N$. Since a smooth curve is irreducible, i.e. has a unique component, the degree of the complete intersection of hypersurfaces which meet in a smooth curve is the product of their degrees. $\square$

3. Bundles

In this section we consider some theory of certain sheaves of modules on a scheme, which we call vector bundles. This discussion has separated into a discussion of features of, computational tools for, and constructions of bundles, and then examples of different bundles. These two topics meet in the discussion of the canonical bundle, and the canonical ring in particular, for a curve. While there are probably treatments of each of the facts in this section in [5], we cite a variety of sources instead, both for readability as well as proximity to the overarching problem of the genus formula.

Formally, we consider the following kinds of sheaves of modules throughout the rest of these notes.

Definition 3.1. Let $X$ be a curve over $\mathbb{C}$. Then a vector bundle of rank $n$ on $X$ is a locally free sheaf of rank $n$ $\mathcal{O}_X$-modules. A line bundle on $X$ is a vector bundle of rank 1.

To make the abstract notion of a sheaf such as a line bundle more convenient for computation, we will often use the following notion of divisors in place of line bundles. Indeed in many situations, such as the case of a smooth curve, there is a correspondence between line bundles and divisors.
Definition 3.2. Let $X$ be a scheme of dimension $n$ over $\mathbb{C}$. Then a **divisor** on $X$ is a formal sum of codimension 1 subschemes of $X$.

Definition 3.3. Let $X$ be a scheme of dimension $n$ over $\mathbb{C}$ with function field $\kappa(X)$. We say that two divisors $D$ and $E$ on $X$ are **linearly equivalent** if there is some $f \in \kappa(X)$ such that $\text{div}(f) \overset{\text{def}}{=} Z(f) - P(f) = D - E$, where $Z(f)$ and $P(f)$ respectively denote the zeros and poles of $f$, counting multiplicities. Write $\text{Div}(X)$ for the free abelian group of divisors up to linear equivalence on $X$.

**Example.** When $X \subset \mathbb{P}^N$ is a curve, a divisor on $X$ is a formal sum of points on $X$.

**Theorem 3.4.** The categories of divisors up to linear equivalence and line bundles up to isomorphism, both on a smooth, projective variety, are equivalent.

**Remark.** Since in these notes we consider the particular case when $X$ is a smooth curve, we will conflate line bundles and divisors on $X$. When $X$ is not smooth, only a strict subset of divisors, known as Cartier divisors, correspond to line bundles. When the hypothesis of smoothness is relevant, we will denote the missing assumption that the corresponding divisors in question are Cartier with parenthesis.

Without worrying about well-definedness or keeping track of equivalences, we can naively spell out the correspondence between line bundles and divisors quite neatly. Given $L$ a line bundle on $X$ and $s$ a rational section of $L$, the associated divisor is

$$\text{div}(s) = Z(s) - P(s) \in \text{Div}(X).$$

Conversely, given $D = \sum n_i P_i$ a (Cartier) divisor on $X$, $\mathcal{O}_X(D)$ is a line bundle on $X$, where

$$\mathcal{O}_X(D) = \{ f \in \kappa(X) : f \text{ has a poles at worst } D \},$$

where $\kappa(X)$ is the function field of $X$, so if locally $f = \frac{g}{h}$, then $\# Z(h) \leq \text{deg}(D)$, where $\text{deg}(D) = \sum n_i$.

3.1. **Facts about Bundles.** Line bundles can define rational maps to projective space.

**Definition 3.5.** Let $X$ be a scheme over $\mathbb{C}$ and let $L$ be a line bundle on $X$. Suppose $s_0, \cdots, s_r$ is a basis for $H^0(X, L)$. Then there exists a rational map

$$\varphi_L : X - \{ s_0 = \cdots = s_r = 0 \} \rightarrow \mathbb{P}^r$$

given by $P \mapsto (s_i(P))_{i=0}^r$.

**Remark.** This is a rational map in the sense that it is defined only on a dense open subset of $X$ rather than the full space. In particular if $L$ is a basepoint-free line bundle, i.e. $\{ s_0 = \cdots s_n = 0 \} = \emptyset$, then $\varphi_L$ may define a map to $\mathbb{P}^r$ on all of $X$.

We need a sort of composite definition to make these induced maps from line bundles somehow preserve the geometry of the scheme they are defined on.

**Definition 3.6.** Say the line bundle $L$ is **ample** if there is some nonnegative $r \in \mathbb{Z}$ such that $L^\otimes r$ is very ample. Say a line bundle $L$ is **very ample** if the embedding $\varphi_L : X \rightarrow \mathbb{P}^r$ by global sections of $L$ is a closed immersion and $L$ is basepoint free.

Next, we’ll do an a priori ring theoretic construction on sheaves of modules over Proj of a graded ring. This is an extended example of a line bundle which not only lies on the ambient projective scheme which our (embedded) curves live in, but also deals explicitly with hypersurfaces, which cut out our curves as complete intersections.

We start with a fact simply about graded rings.
**Definition 3.7.** [2] Let \( S = \bigoplus_{e \geq 0} S_e \) be a graded ring. The \( d \)th Serre twist of \( S \) is the \( S \)-module \( S(d) \) given by \( S(d)_e \overset{\text{def}}{=} S_{e+d} \).

Now we introduce a sheaf on \( \text{Proj} \) of a graded ring.

**Definition 3.8.** [2] Let \( S \) be a graded ring and let \( M \) be a graded \( S \)-module. Then there is a sheaf of modules \( M \) on \( \text{Proj}(S) \) defined by (the sheafification of)

\[
\hat{M}(D_+(f)) = M\left[\frac{1}{f}\right]_0,
\]

where \( M\left[\frac{1}{f}\right]_0 \) is the 0th graded component of \( M\left[\frac{1}{f}\right] \).

Finally, we relate this sheaf of modules to the structure sheaf.

**Definition 3.9.** [2] Let \( S \) be a graded ring and write \( \mathbb{P}^N_S \) for \( \text{Proj} S[x_0, \cdots, x_N] \) for \( x_i \) indeterminates. The Serre twisting sheaf \( \mathcal{O}_{\mathbb{P}^N_S}(d) \) on \( \mathbb{P}^N_S \), is \( \mathcal{O}_{\mathbb{P}^N_S}(d) \overset{\text{def}}{=} S_{\mathbb{P}^N_S}(d) \).

The following Theorem from [2] is instrumental computing the Picard group of \( \mathbb{P}^N \) as well as making tensor products of line bundles on \( \mathbb{P}^N \) into a problem about elementary addition of degrees.

**Theorem 3.10.** Let \( F \) be a field. For any \( d \in \mathbb{Z}_{\geq 0} \)

\[
\mathcal{O}_{\mathbb{P}^N_F}(d) \cong \mathcal{O}_{\mathbb{P}^N_F}(dH),
\]

for \( H \subset \mathbb{P}^N_F \) any hyperplane.

**Proof.** Let \( S = F[x_0, \cdots, x_N] \) and fix \( d \) a non-negative integer. Recall that \( S(d) \overset{\text{def}}{=} S_{e+d} \) by Definition 3.7, so \( S(d) = \bigoplus_{e \geq 0} F[x_0, \cdots, x_N]_{e+d} \). Let \( \hat{S} \) be the sheaf of \( S \)-modules on \( \text{Proj} S \cong \mathbb{P}^N_F \) given by (sheafifying)

\[
\hat{S}(d)(D_+(f)) \cong S\left[\frac{1}{f}\right]_d = F[x_0, \cdots, x_N, \frac{1}{f}]_d,
\]

where the nontrivial isomorphism of localized rings is from Example 2 on page 708 in [1]. Fix an affine open cover

\[
\mathbb{P}^N_F = \bigcup_{i=0}^N U_i = \bigcup_{i=0}^N D_+(x_i),
\]

and for each affine open \( U_i \) where \( 0 \leq i \leq N \) consider a map \( \varphi_i : \hat{S}(d)(U_i) \to \mathcal{O}_{\mathbb{P}^N_F}(dH)(U_i) \) given by

\[
f \mapsto f(p(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N))
\]

for \( f \in F[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]_d \), where \( p \in S_{N-1} \) is a permutation of indices of coordinates. Note that we might equivalently define our map by a composition of \( f \) with a linear change of basis for homogeneous degree \( d \) polynomials in \( x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N \). In other words, each \( \varphi_i \) is a composition of the identity map on \( F[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]_d \) with an automorphism of \( F[x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N]_0 \), and therefore is a well-defined ring homomorphism. There is a well-defined injective inverse map by composing \( f^{-1} \) with the inverse permutation-of-coordinates or respectively the inverse of the change-of-basis automorphism, i.e. \( \pi^{-1} \circ f^{-1} \), and therefore on each affine open we have an isomorphism. This way we have isomorphisms

\[
\varphi_{ij} \overset{\text{def}}{=} \varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}},
\]

where \( U_{ij} \overset{\text{def}}{=} U_i \cap U_j \). By part 3 of the proof of Theorem II.3.3 in [5] these morphisms glue. \( \square \)

**Corollary 3.11.** \( \text{Pic}(\mathbb{P}^N) \cong \mathbb{Z} \) as groups.

**Proof.** By Theorem 3.10, we see line bundles on \( \mathcal{O}_{\mathbb{P}^N} \) are unique up to degrees. \( \square \)
3.2. Examples of bundles. To develop a theory of a canonical bundle on a curve, and to compute it, we will need four standard kinds of vector bundles which exist on many kinds of schemes. These are the tangent and cotangent sheaves, the normal bundle, and finally the canonical bundle itself.

3.2.1. The Sheaf of Differentials. As usual in this section, we begin with some facts about graded rings.

**Definition 3.12.** Let $A$ be a commutative ring with 1, let $B$ be an $A$-algebra and let $M$ be a $B$-module. An $A$-derivation of $B$ into $M$ is a map $d : B \to M$ such that

1. $d(b + b') = d(b) + d(b')$ for all $b, b' \in B$,
2. $d(bb') = bd(b') + b'd(b)$ for all $b, b' \in B$, and
3. $d(a) = 0$ for all $a \in A$.

Now we may formally define a module of differentials in the “right” way to extend the definition to schemes.

**Definition 3.13.** Let $A$ be a commutative ring with 1 and let $B$ be an $A$-algebra. Define the module of relative differential forms of $B$ over $A$ to be the $B$-module $\Omega_{B/A}$ equipped with the $A$-derivation $d : B \to \Omega_{B/A}$ which satisfies the universal property that for any $B$-module $M$ and any $A$-derivation $d' : B \to M$, there exists a unique $B$-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

**Definition 3.14.** Let $X$ be a scheme of dimension $n$ over $\mathbb{C}$, and let $B \defeq \mathbb{C}[x_0, \ldots, x_n]$. Then $\Omega_{B/\mathbb{C}}$ is the free $B$-module of rank $n$ generated by $dx_0, \ldots, dx_n$, and we denote by $\Omega_X$ the sheaf of differential 1-forms on $X$, with associated module $\Omega_{B/\mathbb{C}}$.

Any actual treatment of duals of sheaves is besides the point in this discussion, so we state a definition of a tangent sheaf so that we can connect differentials and the normal bundle, which we turn to next.

**Definition 3.15.** Let $X$ be a scheme over $\mathbb{C}$. The tangent bundle $T_X$ to $X$ is the bundle $T_X \defeq \Omega_X^\vee$.

3.2.2. Normal Bundle. At first glance, the normal bundle appears to be simply yet another sheaf of modules on a scheme, this time with a particularly unfriendly looking quotient definition. However, we have carefully picked an exceptionally friendly kind of scheme, a complete intersection, to compute the normal bundle for.

Formally, we define the vector bundle of normal vectors to a subscheme as follows.

**Definition 3.16.** Suppose $X \subset Y$ is an inclusion of schemes over $\mathbb{C}$. Then there is an inclusion of bundles $T_X \subset T_Y|X$ and the quotient bundle

$$N_X \defeq T_Y|X/T_X$$

is the normal bundle to $X$ in $Y$.

The connection between determinants of bundles, complete intersections of hypersurfaces, and curves all hinges on the following theorem.

**Theorem 3.17.** Suppose $X \subset \mathbb{P}^N$ is a curve which is the complete intersection of hypersurfaces $D_1, \ldots, D_r \subset \mathbb{P}^N$. Then $N_X = \mathcal{O}_{\mathbb{P}^N}(D_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^N}(D_r)$

3.2.3. The Canonical Bundle. I decline to comment on why the canonical bundle is so named. This definition has two forms: the explicit catch-phrase “top exterior power of the sheaf of differentials” definition for computations, and for experts, the derived functor definition.
Definition 3.18. Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension $n$. We define the canonical bundle $\omega = \omega_X$, a line bundle on $X$, by

$$\omega \overset{\text{def}}{=} \bigwedge^n \Omega^1_X = \operatorname{Ext}^N_{\mathcal{O}_{\mathbb{P}^N}}(\mathcal{O}_X, \omega_{\mathbb{P}^N}),$$

where $N = \dim H^0(X, \omega_X) - 1$, $\Omega^1_X$ is the sheaf of regular differential one-forms on $X$ from Definition 3.13, and $\operatorname{Ext}$ is the derived functor of sheaf $\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^N}}(\cdot, \omega_{\mathbb{P}^N})$ from the Lectures [6].

This line bundle is so special that the associated divisor has a distinguished name.

Definition 3.19. Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension $n$. The canonical divisor $K_X$ on $X$ is the (Cartier) divisor associated to $\omega_X$.

We make one final restriction on the kind of curves which we consider from this point forward.

Definition 3.20. Let $X \subset \mathbb{P}^N$ be a curve and suppose that $\omega_X$ is very ample. Then we call $X$ a canonical curve.

Part of the reason we think of curves $X$ as projective in these notes is so that we can compute the coordinate ring of $X$ by means of the coordinate ring of $\mathbb{P}_C^N = \operatorname{Proj} \mathbb{C}[x_0, \cdots, x_N]$. In particular we can map to this ring by means of the map to $\mathbb{P}^N$ associated to $\omega_X$ given in Definition 3.5.

Definition 3.21. Let $X$ be a scheme over $\mathbb{C}$. The canonical ring of $X$ is the ring

$$R = R(X, \omega_X) = \bigoplus_{n \geq 0} H^0(X, \omega_X^\otimes n).$$

Remark. It is a nontrivial and relatively recent result that the canonical ring is finitely generated, but the proof in full generality is too involved for these notes, which are concerned with the more classical theorem mentioned below.

Remark. For $\mathcal{L}$ any very ample line bundle on a scheme $X$ over $\mathbb{C}$, we can define a section ring of $\mathcal{L}$ analogously to the canonical ring defined above. One particularly relevant example for these notes is the Arbarello-Sernesi module of $X$ and $\mathcal{L}$ a line bundle on $X$ which is the graded module

$$\bigoplus_{q \in \mathbb{Z}} H^0(X, K_X \otimes q\mathcal{L}),$$

which can be used, as in [4], with $\mathcal{L} = K_X$, to prove the Theorem of Enriques, Babbage and Petri, known as Petri’s theorem, that genus 5 curves are the complete intersections of 3 quadrics in $\mathbb{P}^4$, and generalizations of this result for higher genus canonical curves.

Now that we have a basic sense of what a canonical bundle is, and why we care about such a bundle, we turn the discussion to computing it, in the case of curves which are complete intersections of hypersurfaces.

Lemma 3.22. [7] Shafarevich’s Lemma] Let $X$ be a purely $n$-dimensional, non-singular, smooth, projective, algebraic variety over $\mathbb{C}$. Locally, the canonical bundle on $X$ has form $\omega = f(dx \wedge dy)^n$, where $x, y$ are some local parameters and $f$ is some regular function.

We first compute the canonical bundle on the scheme $\mathbb{P}^N = \operatorname{Proj} \mathbb{C}[x_0, \cdots, x_N]$.

Theorem 3.23. $\omega_{\mathbb{P}^N} \cong \mathcal{O}_{\mathbb{P}^N}(-N - 1)$.

Proof. For readability this proof is restricted to the case when $N = 2$. Let $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2]$ and consider some charts

$$U_0 = \{x_0 \neq 0\} \text{ coordinates } (u_1, u_2), \quad \begin{cases} u_1 & \overset{\text{def}}{=} \frac{x_1}{x_0} \smallskip \vspace{-3mm} \\ u_2 & \overset{\text{def}}{=} \frac{x_2}{x_0} \smallskip \vspace{-3mm} \end{cases}$$

$$U_1 = \{x_1 \neq 0\} \quad \text{----} \quad (v_0, v_2), \quad \begin{cases} v_0 & \overset{\text{def}}{=} \frac{x_0}{x_1} \smallskip \vspace{-3mm} \\ v_1 & \overset{\text{def}}{=} \frac{x_1}{x_1} \smallskip \vspace{-3mm} \end{cases}$$

$$U_2 = \{x_2 \neq 0\} \quad \text{----} \quad (w_0, w_1), \quad \begin{cases} w_0 & \overset{\text{def}}{=} \frac{x_1}{x_2} \smallskip \vspace{-3mm} \\ w_1 & \overset{\text{def}}{=} \frac{x_2}{x_2} \smallskip \vspace{-3mm} \end{cases}$$
By Shafarevich’s Lemma 3.22 sections of $\omega_{\mathbb{P}^2}$ over $U_0$ have form $f(u_1, u_2)du_1 \wedge du_2$ for some $f \in \mathcal{O}_{\mathbb{P}^2}$, so consider the section $du_1 \wedge du_2$ in particular. Away from $U_0$ there is one location in $\mathbb{P}^2$ where we want to make sense of our section $du_1 \wedge du_2$, namely the divisor $x_0 = 0$. In the coordinates of the chart $U_1$, which contains the divisor $x_0 = 0$, we observe with some elementary calculus that

$$du_1 \wedge du_2 = \left(\frac{-1}{u_0^2} du_0\right) \wedge \left(\frac{u_0 du_2 - u_2 du_0}{u_0^3}\right),$$

and since $e_i \wedge e_i \overset{\text{def}}{=} 0$ for any vector $e_i$, we conclude

$$du_1 \wedge du_2 = \frac{-1}{u_0^3} du_0 \wedge du_2.$$

Since $\frac{-1}{u_0}$ has a pole of order 3 on $u_0 = 0$ as desired we are done.

We will use a version of the adjunction formula presented in [2] to compute the canonical bundle of our complete intersections.

**Theorem 3.24.** [2] If $X \subset \mathbb{P}^N$ is a smooth subscheme with normal bundle $N_X$ then

$$\omega_X \cong \omega_{\mathbb{P}^N}|X \otimes \det(N_X).$$

4. Main Result

**Theorem 4.1.** Let $X \subset \mathbb{P}^4$ be the complete intersection of smooth degree $d_1, d_2, d_3$ hypersurfaces $D_1, D_2$ and $D_3 \subset \mathbb{P}^4$. Then $X$ is a curve of genus

$$g = \frac{(d_1 + d_2 + d_3 - 5)d_1d_2d_3 - 2}{2}.$$

**Proof.** Recall that by Exercise I.2.17.b in [5], $X$ is a set-theoretic complete intersection and therefore a curve since it is the intersection of 3 hypersurfaces in $\mathbb{P}^4$, i.e. a variety of dimension 1 per Definition 2.3. By the Adjunction formula 3.24 we compute

$$\omega_X = \omega_{\mathbb{P}^4}|X \otimes \det(N_X).$$

Using Theorem 3.17 and Theorem 3.23 we see

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(-5)|X \otimes \det [\mathcal{O}_{\mathbb{P}^4}(D_1)|X \oplus \mathcal{O}_{\mathbb{P}^4}(D_2)|X \oplus \mathcal{O}_{\mathbb{P}^4}(D_3)|X].$$

We can compute the determinant with Theorem 1.5 and since the $\mathcal{O}_{\mathbb{P}^4}(D_i)|X$ are line bundles for $1 \leq i \leq 3$, Definition 1.4 becomes $\det(\mathcal{O}_{\mathbb{P}^4}(D_i)|X) = \Lambda^3 \mathcal{O}_{\mathbb{P}^4}(D_i)|X = \mathcal{O}_{\mathbb{P}^4}(D_i)|X$ for each $i$, so we get

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(-5)|X \otimes \mathcal{O}_{\mathbb{P}^4}(D_1)|X \otimes \mathcal{O}_{\mathbb{P}^4}(D_2)|X \otimes \mathcal{O}_{\mathbb{P}^4}(D_3)|X$$

by Theorem 1.5. Since $\text{Pic}(\mathbb{P}^N) \cong \mathbb{Z}$ so line bundles are unique up to degrees, using Serre twist notation 3.10 and the fact that restrictions commute with tensors since restriction is a right adjoint functor, we rewrite

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(-5) \otimes \mathcal{O}_{\mathbb{P}^4}(d_1) \otimes \mathcal{O}_{\mathbb{P}^4}(d_2) \otimes \mathcal{O}_{\mathbb{P}^4}(d_3)|X.$$

Finally, making use of the convenient notation choice above and the Picard group again,

$$\omega_X = \mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5)|X = \mathcal{O}_X(d_1 + d_2 + d_3 - 5).$$

By Theorem 3.10 we have an isomorphism

$$\mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5)|X \cong \mathcal{O}_{\mathbb{P}^4}((d_1 + d_2 + d_3 - 5)H)|X$$

for $H$ any hyperplane divisor. Being a hyperplane divisor, $H$ will intersect $X$, which has degree $d_1d_2d_3$ by Corollary 2.5, exactly $\deg(X) = d_1d_2d_3$ times, so that

$$\deg(\mathcal{O}_X(d_1 + d_2 + d_3 - 5)) = (d_1 + d_2 + d_3 - 5)d_1d_2d_3.$$
By Riemann-Roch and Corollary 2.5 we compute 
\[ \deg(K_X) = \deg(\mathcal{O}_{\mathbb{P}^4}(d_1 + d_2 + d_3 - 5)|_X) = 2g - 2 \]
\[ (d_1 + d_2 + d_3 - 5)d_1d_2d_3 = 2g - 2, \]
so 
\[ g = \frac{(d_1 + d_2 + d_3 - 5)d_1d_2d_3 - 2}{2}. \] (4.1)

\[ \square \]

**Corollary 4.2.** The complete intersection of 3 distinct smooth quadrics in \( \mathbb{P}^4 \) is a curve of genus 5.

**Proof.** For each \( i \), we have \( d_i = 2 \) and so by our formula 4.1 we compute \( g = 5 \). \( \square \)
References


