The Geometry of Drinfeld Modular Forms

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Classical Theory

The classical (number-field) number theory and algebraic geometry which we wish to study in the Drinfeld setting is based around the following objects:

1. Modular forms - functions $f : \mathcal{H} \to \mathbb{C}$ such that $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $\mathcal{H}$ is the complex upper half-plane, $k \in \mathbb{Z}_{\geqslant 0}$, and $\Gamma \leqslant \mathrm{PSL}_2(\mathbb{R})$. Note that $f(\gamma z) d(\gamma z)^{\otimes k/2} = f dz^{\otimes k/2}$. Let $M_k(\Gamma)$ denote the $\mathbb{C}$-vector space of modular forms of weight k for Γ and $X = \Gamma \backslash \mathcal{H}^*$ denote the projective modular curve associated to Γ. Then
$$M_k(\Gamma) \xrightarrow{\sim} H^0(X, \Omega^1_X(\Delta)^{\otimes k/2})$$
$$f \mapsto f dz^{\otimes k/2}$$
where Δ denotes the log divisor of cusps of Γ.
2. Modular curves - tame Deligne-Mumford stacks $\mathscr{X}$ which are the moduli of elliptic curves with certain level structures. Define such an $\mathscr{X}$ to be the algebraization of the compactified orbifold quotient $X = \Gamma \backslash \mathcal{H}^*$.
3. Section rings - for X a curve (scheme or stack) over $\mathbb{C}$ and $\mathscr{L}$ a line bundle on X, the section ring of $\mathscr{L}$ is
$$R(X, \mathscr{L}) = \bigoplus_{d \geqslant 0} H^0(X, \mathscr{L}^{\otimes d}).$$
example: For $\mathscr{X} = \mathscr{X}(\Gamma)$ as above, we have $R(\mathscr{X}, \Delta) = \bigoplus_{k \in 2\mathbb{Z}_{\geqslant 0}} M_k(\Gamma)$, induced by the isomorphisms above.

Drinfeld Setting

Let q be a power of an odd prime and T an indeterminate. The Drinfeld, or function field, setting may be introduced by the following analogies with the classical setting:

$\mathbb{Z}$	$A = \mathbb{F}_q[T]$
$\mathbb{Q}$	$K = \mathbb{F}_q(T)$
$\mathbb{R}$	$K_\infty = \mathbb{F}_q(1/T)$
$\mathbb{C}$	$C = \widehat{\overline{K_\infty}}$
$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$	$\Omega = C - K_\infty$
$e^{2\pi i z}$	$u(z) \overset{def}{=} \bar{\pi}^{-1} \sum_{a \in A} \frac{1}{z+a}$, for fixed $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$ (Carlitz period)
$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{H}$	$\mathrm{GL}_2(A) \backslash \Omega$
$\bigoplus_{k \in 2\mathbb{Z}_{\geqslant 0}} M_k(\Gamma)$	$\bigoplus_{k \geqslant 0,\ l \pmod{q-1}} M_{k,l}(\Gamma)$

Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup. A (Drinfeld) modular form of weight $k \in \mathbb{Z}_+$ and type $l \in \mathbb{Z}/((q-1)\mathbb{Z})$ is a holomorphic function $f : \Omega \to C$ such that

1. $f(\gamma z) = \det(\gamma)^{-l}(cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & b \end{pmatrix} \in \Gamma$, and
2. f is holomorphic at the cusps of Γ.

Theorem (Drinfeld):
There exists a smooth, irreducible, affine algebraic curve Y_Γ over C called a Drinfeld modular curve, such that $\Gamma \backslash \Omega$ and the underlying (rigid) analytic space Y_Γ^{an} of Y_Γ are canonically isomorphic as rigid analytic spaces over C.
A smooth projective model X_Γ for the affine algebraic Drinfeld modular curve Y_Γ is the coarse space of $\mathscr{X}_\Gamma$, the moduli stack of rank 2 Drinfeld modules with Γ level structure.

Existing Results

In his 1986 monograph Gekeler asks for a description of the algebras of Drinfeld modular forms in terms of generators and relations. The only examples of results in this direction so far are:

- **Gekeler/Goss** - $M(\mathrm{GL}_2(A)) = C[g, h]$
- **Cornelissen** - the algebra of modular forms for $\Gamma(\alpha T + \beta)$;
- **Dalal/Kumar** - the algebra of modular forms for $\Gamma_0(T)$;
- **Armana** - for any level $N \in A$ there is an isomorphism
$$M^2_{2,1}(\Gamma_0(N)) \xrightarrow{\sim} H^0(X_0(N)^{\mathrm{an}}, \Omega^1_{\mathrm{an}})$$
$$f \mapsto \overline{\pi}^{-1} f dz.$$

How Our Theory Works

- Throughout we suppose that our congruence subgroup $\Gamma \leqslant \mathrm{GL}_2(A)$ contains the matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}$ for all $\alpha, \alpha' \in \mathbb{F}_q^\times$. This means that if $f \in M_{k,l}(\Gamma)$, we have
$$f\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} z\right) = f\left(\frac{\alpha z}{\alpha}\right) = \alpha^{k-2l} f(z) = f(z),$$
i.e. if $M_{k,l}(\Gamma) \neq 0$, then $k \equiv 2l \pmod{q-1}$.
- For $\overline{\pi}$ the Carlitz period and $u(z)$ the parameter at ∞, we have
$$\frac{du}{u^2} = -\overline{\pi} dz,$$
so dz has a double pole at the cusps and $dz^{\otimes k/2}$ has a pole of order $k = \frac{k}{2}(2)$. We observe that for f a modular form, the differentials $f dz^{\otimes k/2}$ may have at worst poles of order k at the cusps of a Drinfeld modular curve.

Where Old Ideas Stop Working

1. **Cusps of a Drinfeld modular curve**: The cusps, orbits $\Gamma \backslash \mathbb{P}^1(K)$, may be stacky points on a Drinfeld modular curve $\mathscr{X}_\Gamma$ since they are stabilized by the group of upper triangular matrices in Γ. This means the log divisor of cusps of a stacky Drinfeld modular curve is supported at stacky points, which changes important Riemann-Roch calculations. Currently, techniques to compute canonical rings of log stacky curves with stacky log divisors are only known for genus 0, but in joint work with Mike Cerchia and Evan O'Dorney, we hope to extend these to genus 1 in work appearing soon.
2. **Modular Forms Do Not Descend to the Modular Curve**: If $f \in M_{k,l}(\Gamma)$ then $f(dz)^{\otimes k/2}$ is not Γ-invariant:
$$f(\gamma z) d(\gamma z)^{\otimes k/2} = (cz+d)^k (\det \gamma)^{-l} \frac{\det \gamma^{k/2}}{(cz+d)^k} f(z) dz^{\otimes k/2}$$
$$= (\det \gamma)^{\otimes l - k/2} f dz^{\otimes k/2},$$
where, as $\det \gamma \in \mathbb{F}_q^\times$, it need not be that $(\det \gamma)^{\otimes l-k/2} = 1$, unless $\det \gamma$ is the square of some element in $\mathbb{F}_q^\times$.
3. **Rigid GAGA for Stacks** We need to translate sheaves from rigid analytic spaces to affine algebraic schemes with rigid GAGA, then to sheaves on (tame, separably rooted Deligne-Mumford) stacky curves. We need notions of a rigid analytic stack, and GAGA theorems for rigid analytic and algebraic stacks.

Main Theorem

Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$ and such that $\det(\gamma)$ is a square element in $\mathbb{F}_q^\times$ for all $\gamma \in \Gamma$. Let Δ be the log divisor of cusps of the Drinfeld modular curve $\mathscr{X} = \mathscr{X}_\Gamma$.

Drinfeld modular forms for Γ are differentials on $(\mathscr{X}, 2\Delta)$:

There is an isomorphism of graded rings
$$M(\Gamma) \cong R(\mathscr{X}, \Omega^1_{\mathscr{X}}(2\Delta)),$$
where $\Omega^1_{\mathscr{X}}$ is the sheaf of differentials on $\mathscr{X}$. The isomorphism is given by isomorphisms
$$M_{k,l}(\Gamma) \xrightarrow{\sim} H^0(\mathscr{X}, \Omega^1_{\mathscr{X}}(2\Delta)^{\otimes k/2})$$
of components given by $f \mapsto f(dz)^{\otimes k/2}$.
Remark: If l_1, l_2 are the two solutions to $k \equiv 2l \pmod{q-1}$, then we have $M_{k,l_1}(\Gamma) = M_{k,l_2}(\Gamma)$.

Main Theorem

Let $\Gamma \leqslant \mathrm{GL}_2(A)$ be a congruence subgroup containing the diagonal matrices in $\mathrm{GL}_2(A)$. Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$. Then
$$M(\Gamma) \cong M(\Gamma_2),$$
with
$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$
on each graded piece, where l_1, l_2 are the two solutions to $k \equiv 2l \pmod{q-1}$.

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