Computing the Canonical Ring of Certain Stacks

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Notation

q - a power of an odd prime.
K - the function field of some smooth, connected, projective curve over a field of size q, e.g. $\mathbb{P}^1$

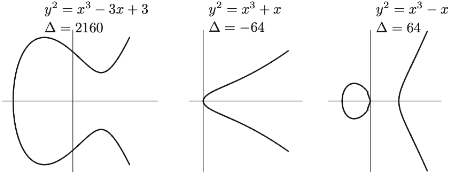
Classical Setting	Function Field
$\mathbb{Z}$	$A \overset{def}{=} \mathbb{F}_q[T]$
$\mathbb{Q}$	$K \overset{def}{=} \mathbb{F}_q(T)$
$\mathbb{R}$	$K_\infty \overset{def}{=} \mathbb{F}_q\left(\left(\frac{1}{T}\right)\right)$
$\mathbb{C}$	$C \overset{def}{=} \widehat{\overline{K_\infty}}$
$\mathcal{H} = \{a + bi \in \mathbb{C} : b > 0\}$	$\Omega \overset{def}{=} C - K_\infty$
$\mathrm{SL}_2(\mathbb{Z}) \setminus \mathcal{H}$	$\mathrm{GL}_2(A) \setminus \Omega$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

Elliptic Curves

An **elliptic curve** is (analytically) a torus/$\mathbb{C}$, i.e. a lattice quotient $\mathbb{C}/(\mathbb{Z}z+\mathbb{Z})$ for $z \in \mathcal{H}$;

or (algebraically) a curve defined by:
$E : y^2 = x^3 + A(z)x + B(z)$



[Sil09, Figure 3.1]

Drinfeld Modules

Consider the rank 2 lattice $\Lambda_z = \overline{\pi}(zA + A) \subset C$.The associated **Drinfeld module of rank** 2 is given by

$$\varphi^z(T) = TX + g(z)X^q + \Delta(z)X^{q^2},$$

the image of a ring homomorphism $\varphi^z : A \to C\{X^q\}$,

($C\{X^q\}$ is the non-commutative ring of $\mathbb{F}_q$-linear polynomials/C.)

The classical thing we want to analogize

Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group with finite coarea. Let $\mathscr{X}(\Gamma)$ denote the stacky curve over $\mathbb{C}$ which is the algebraization of the compactified orbifold quotient $X = \Gamma \setminus \mathcal{H}^{(*)}$. We know (e.g. [VZB22, Chapter 6])

$$M(\Gamma) \overset{def}{:=} \bigoplus_{k \geq 0} M_k(\Gamma) \xrightarrow{\sim} \bigoplus_{k \geq 0} H^0(\mathscr{X}_\Gamma, \Omega^1_{\mathscr{X}_\Gamma}(\Delta)^{\otimes k/2}) \overset{def}{=:} R(\mathscr{X}_\Gamma; \Delta),$$

$$f \mapsto f dz^{\otimes k/2}$$

[Gek86, page 13]:

in § 4. It would be desirable to have a description by generators and relations, where the generating modular forms should have an elementary interpretation by means of Drinfeld modules. In § 5, the genera of

Why Stacks? What are Stacks?

Modular forms are *always* sections of a line bundle.
However,

$$H^0(X, L^{\otimes k}) \neq M_k(\Gamma) \qquad \text{and} \qquad R(X; L) \neq M(\Gamma),$$

where

$$\begin{cases} X & = \text{moduli scheme,} \\ L & = \text{appropriate line bundle,} \\ M & = \text{vector space of modular forms.} \end{cases}$$

So, **what are stacks?**

1-category	2-category
functor / pre-sheaf	fibered category
separated pre-sheaf	pre-stack
sheaf	stack
algebraic space / scheme	algebraic stack
variety	algebraic stack of finite type over a field

So, what are stacks?

Definition

A **stacky curve** over an algebraically closed field $\mathbb{K}$ is:

- a smooth, integral, proper, scheme X of dimension 1, together with
- a finite number of closed points $P_1, \ldots, P_r$ called **stacky points** with stabilizer orders $e_1, \ldots, e_r \in \mathbb{Z}_{\geq 2}$.

Example ([Lau96, Corollary 1.4.3])

The moduli space $\mathcal{M}_A^2$ of rank 2 Drinfeld modules with no level structure is known to be a Deligne-Mumford algebraic stack of finite type over $\mathbb{F}_p$.

Stacky Curves 101

Let $\mathscr{X}$ denote a stacky curve with **signature** $\sigma = (g; e_1, \ldots, e_r)$. We say that $D \in \mathrm{Div}(\mathscr{X})$ has

$$\deg(D) = \deg\lfloor D \rfloor = \deg \left\lfloor \sum_i a_i P_i \right\rfloor \overset{def}{=} \sum_i \lfloor a_i \rfloor \pi(P_i),$$

where $\pi : \mathscr{X} \to X$ is the coarse space morphism. The **(log) canonical ring of** $(\mathscr{X}; \Delta)$ is

$$R(\mathscr{X}; \Delta) = \bigoplus_{d \geq 0} H^0(\mathscr{X}, d(K_{\mathscr{X}} + \Delta)),$$

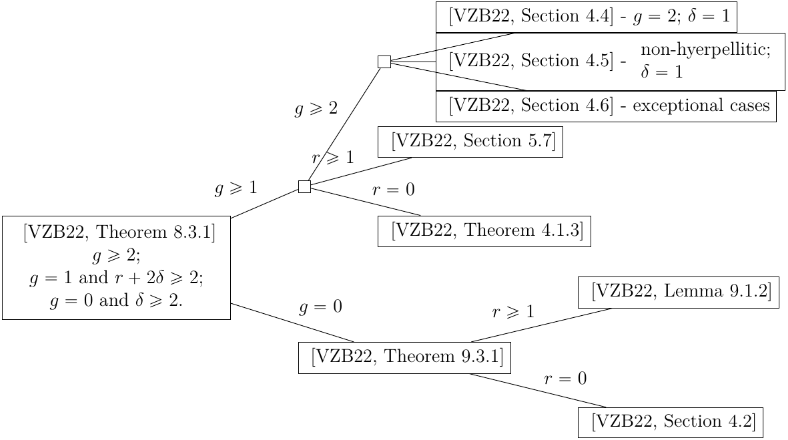
where

$$K_{\mathscr{X}} \sim K_X + \left(\sum_{i=1}^{r} \left(1 - \frac{1}{e_i}\right) P_i \right),$$

is a **canonical divisor** of $\mathscr{X}$ and $\Delta = \sum_j Q_j \in \mathrm{Div}(\mathscr{X})$ is a **log divisor**.

Computing the Canonical Ring of a Stacky Curve

[VZB22] gives an inductive presentation of $R(\mathscr{X})$ for $\mathscr{X}$ with $\sigma = (g; e_1, \ldots, e_r)$ in terms of $R(\mathscr{X}')$ with $\sigma' = (g; e_1, \ldots, e_{r-1})$:



Example of (an inductive) presentation of section rings

Example ([CFO24, Example 5.1])

Let X denote a genus 1 curve over some field $\Bbbk$.

[VZB22, Example 5.7.7] Let $D' = \frac{1}{2}P_1 + \frac{1}{2}P_2$

[VZB22, Example 5.7.9] Let $D = D' + \frac{1}{2}P_3 = \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{1}{2}P_3$.

By the Generalized Max Noether Theorem [VZB22, Lemma 3.1.4],

$$H^0(X, 2D) \otimes H^0(X, (d-2)D) \to H^0(X, dD)$$

is surjective for $d > 5$, so all generators occur in degree < 5.
The minimal presentations have the form

$$R_D = \Bbbk[u, x_1, x_2]/I_D$$
$$R_{D'} = \Bbbk[u, x_1, x_2^2]/I_{D'},$$

where I_D, $I_{D'}$ are the relation ideals. In particular, R_D is generated over $R_{D'}$ by x_2.

Old Friends

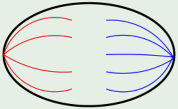
Example (Goss and Gekeler's famous $\mathrm{GL}_2(A)$-forms)

- g of weight $q-1$ and type 0,
- Δ of weight q^2-1 and type 0,
- h of weight $q+1$ and type 1.

$$\bigoplus_{k\geq 0} M_{k,0}(\mathrm{GL}_2(A)) = C[g,\Delta] \quad \text{and} \quad \bigoplus_{\substack{k\geq 0 \\ l \pmod{q-1}}} M_{k,l}(\mathrm{GL}_2(A)) = C[g,h].$$

Example (Stacky j-line)

$\mathscr{X}_{\mathrm{GL}_2(A)} \cong \mathbb{P}^1(q-1,q+1)$ is a **football** (see e.g. [VZB22, 5.3.14]):



But, $R(\mathscr{X}_{\mathrm{GL}_2(A)}) \neq C[g,h]$.

What goes “Wrong” in Function Fields

Among other resources, we have:

$$\begin{cases} \text{[Gek01]} & \text{for signatures of Drinfeld modular curves, and} \\ \text{[VZB22]} & \text{for computing canonical rings of stacky curves.} \end{cases}$$

So, **where's our modular forms = sections of a line bundle?**
We will consider:

- weight and type of Drinfeld modular forms;
- exponentials and u-series;
- special congruence groups $\Gamma \leq \mathrm{GL}_2(A)$;
- elliptic points and cusps of Drinfeld modular curves;
- GAGA for rigid analytic stacks.

Drinfeld Modular Forms

Definition

Let $\Gamma \leq \mathrm{GL}_2(A)$ be a congruence subgroup. A **modular form** of **weight** $k \in \mathbb{Z}_{\geq 0}$ and **type** $l \in \mathbb{Z}/((q-1)\mathbb{Z})$ is a map $f : \Omega \to C$ such that

1. f is holomorphic on Ω and at the cusps of Γ;
2. $f(\gamma z) = \det(\gamma)^{-l}(cz+d)^k f(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Lemma ([Gek88, Remark (5.8.i)])

If $M_{k,l}(\Gamma) \neq 0$*, then* $k \equiv 2l \pmod{q-1}$.

Proof.

If f is non-zero modular for Γ of weight k and type l then

$$f(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right) z) = f\left(\frac{\alpha z}{\alpha}\right) = f(z) = \alpha^k \alpha^{-2l} f(z).$$

□

"Fourier series" for Drinfeld Modular Forms

Definition

We define a **parameter at infinity**

$$u(z) \overset{def}{=} \frac{1}{e_{\bar{\pi}A}(\bar{\pi}z)} = \frac{1}{\bar{\pi}e_A(z)} = \bar{\pi}^{-1}\sum_{a\in A}\frac{1}{z+a}.$$

Recall:

- $u(\alpha z) = \alpha^{-1}u(z)$ for any $\alpha \in \mathbb{F}_q^\times$.
- u-series coefficients for a Drinfeld modular form uniquely determine the form.

Lemma

$\frac{de_A(z)}{dz} = 1 \Rightarrow \frac{du}{u^2} = -\bar{\pi}dz$, *i.e. the differential* dz *has a double pole at* ∞.

Special Congruence Subgroups

Drinfeld modular forms are *sensitive to determinants*, so consider some "friendlier" modular forms for Breuer and Böckle's special congruence subgroups:

[Bre16] Let $\Gamma_2 \stackrel{def}{=} \{\gamma \in \Gamma : \det(\gamma) \in (\det \Gamma)^2\}$.
(Suppose $\det \Gamma_2 = (\mathbb{F}_q^\times)^2$.)

[Böckle] Let $\Gamma_1 \stackrel{def}{=} \{\gamma \in \Gamma : \det(\gamma) = 1\}$. Suppose Γ' is such that $\Gamma_1 \leq \Gamma' \leq \Gamma$.

The subgroups Γ_2 and Γ' may be thought of as the inverse image under $\det : \mathrm{GL}_2(A) \to \mathbb{F}_q^\times$ of some subgroup of $\mathbb{F}_q^\times$.

Cusps and Elliptic Points

Let $\Gamma \leq \mathrm{GL}_2(A)$ be a congruence subgroup. Let $X_\Gamma^{\mathrm{an}} = \Gamma \setminus (\Omega \cup \mathbb{P}^1(K))$.

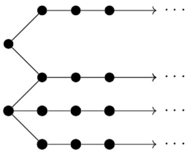
Definition

A **cusp of X_Γ^{an}** is a representative for some orbit $\Gamma \setminus \mathbb{P}^1(K)$. A point $e \in X_\Gamma^{\mathrm{an}}$ is an **elliptic point for Γ** if $\mathrm{Stab}_\Gamma(e)$ is strictly larger than: $\mathbb{F}_q^\times \cong \left\{ \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix} \right) : \alpha \in \mathbb{F}_q^\times \right\}$.

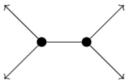
Example (with thanks to Mihran)

Suppose $x \neq y \in A$ have $\deg(x) = 1 = \deg(y)$. Consider $\Gamma_0(xy) \setminus \mathscr{T}$:

The half-line $\mathrm{GL}_2(A)\backslash\mathscr{T}$ (or $\mathrm{SL}_2(A)\backslash\mathscr{T}$)



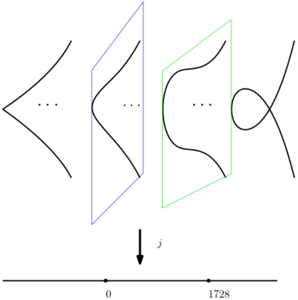
[GN95] computes $\Gamma_0(xy)\backslash\mathscr{T}$ "layer by layer"



We can "read off" that $\mathscr{X}_{\Gamma_0(xy)}$ has 4 cusps.

Cusps are Elliptic Points

Let $\Gamma^1 \leq \mathrm{SL}_2(\mathbb{Z})$. Consider a cartoon of $\Gamma^1 \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$:



$$\Gamma^1 \backslash \mathbb{P}^1(\mathbb{Q}) \leftrightarrow \left(\begin{array}{c} \text{singular} \\ \text{elliptic curves} \end{array} \right),$$

but only elliptic curves with $j = 0$ or 1728 have extra automorphisms.

Let $\Gamma \leq \mathrm{GL}_2(A)$. Consider the moduli $\mathscr{X}_\Gamma = [X_\Gamma / Z(GL_2(A))]$:

$$\mathrm{Aut}(\varphi) \cong \mathbb{F}_q^\times \qquad //\mathbb{F}_q^\times;$$

$$\mathrm{Aut}(\varphi_{(j=0)}) \cong \mathbb{F}_{q^2}^\times \qquad //\mathbb{F}_q^\times;$$

$$\mathrm{Aut}(\varphi_{(j=\infty)}) \cong \{ \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix} \right) \} \qquad //\mathbb{F}_q^\times;$$

so cusps on a stacky Drinfeld modular curve are elliptic points!

Isotropy Groups of Cusps (1/2)

Moduli Interpretation

$$\Gamma \setminus \mathbb{P}^1(K) \leftrightarrow \begin{pmatrix} \text{“degenerate”} \\ \text{Drinfeld modules} \\ \text{of rank 2} \end{pmatrix}, \\ = \begin{pmatrix} \text{Drinfeld modules} \\ \text{of rank 1} \end{pmatrix}$$

Carlitz module:

$$\rho(T) = TX + X^q \leftrightsquigarrow \overline{\pi}A \subset \Omega,$$

where $\overline{\pi} \in K_\infty(\sqrt[q-1]{-T})$.

$$\mathrm{Aut}(\rho) \cong \mathbb{F}_q^\times,$$

“extra” automorphisms specify $\overline{\pi}$.

Gekeler’s Isotropy

It is well-known that ∞ (resp. any cusp of Γ) is stabilized by matrices of form:

$$\left\{ \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma \right\},$$

which is an *infinite* group.

Question: where does this infinite group of translations $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$ go?

Isotropy Groups of Cusps (2/2)

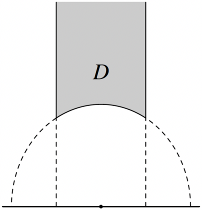


Figure 2.3. The fundamental domain for $SL_2(\mathbf{Z})$

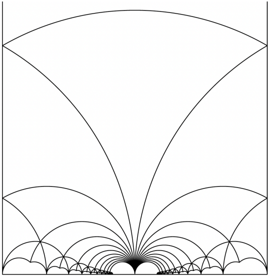
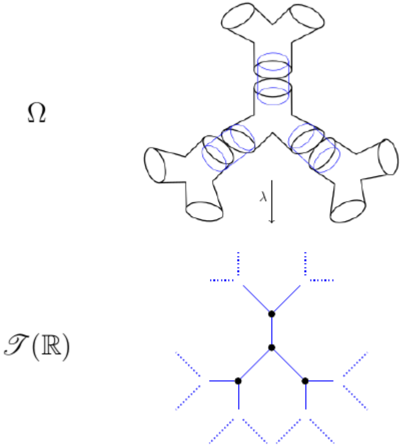


Figure 2.4. Some $SL_2(\mathbf{Z})$-translates of $\mathcal{D}$



Elliptic Points on Stacky Curves

Example (Classical j-line)

- $X(1) = \mathrm{SL}_2(\mathbb{Z}) \backslash (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))$ - the "usual" j-line $\mathbb{P}^1(\mathbb{C})$
- $\overline{\mathcal{M}_{1,1}}$ - DM stack representing the moduli of stable elliptic curves

$\overline{\mathcal{M}_{1,1}}$ is a μ_2-gerbe over $\mathscr{X}(1) = [X(1)/Z(\mathrm{SL}_2(\mathbb{Z}))]$, i.e. $\mathscr{X}(1)$ is a rigidification $\overline{\mathcal{M}_{1,1}}//\mu_2$:

$$\overline{\mathcal{M}_{1,1}} \xrightarrow{\pi} \mathscr{X}(1) \to X(1)$$

$$\mathbb{P}^1(4,6) \xrightarrow{\pi} \mathbb{P}^1(2,3) \to \mathbb{P}^1(\mathbb{C}) \ .$$

Example (Drinfeld j-line)

- $X(1) = \mathrm{GL}_2(A) \backslash (\Omega \cup \mathbb{P}^1(K))$ - the "usual" j-line $\mathbb{P}^1(C)$
- $\overline{\mathcal{M}^2_A}$ - (DM stack) moduli of stable rank 2 Drinfeld modules (no level structure)

$\overline{\mathcal{M}^2_A}$ is a μ_{q-1}-gerbe over $\mathscr{X}(1) = [X(1)/Z(\mathrm{GL}_2(A))]$, i.e. $\mathscr{X}(1)$ is a rigidification $\overline{\mathcal{M}^2_A}//\mu_{q-1}$:

$$\overline{\mathcal{M}^2_A} \xrightarrow{\pi} \mathscr{X}(1) \to X(1)$$

$$\mathbb{P}^1((q-1)^2, q^2-1) \xrightarrow{\pi} \\ \to \mathbb{P}^1(q-1, q+1) \to \mathbb{P}^1(C) \ .$$

Rigid Stacky GAGA

Theorem

Let A be a k-affinoid algebra, for k some non-achimedean field.

- *([PY16, Lemma 7.2]) Let $\mathscr{X}$ be an algebraic stack locally of finite presentation over* $\mathrm{Spec}(A)$. *Suppose that for $\mathcal{F} \in \mathcal{O}_{\mathscr{X}} - \mathrm{Mod}$ we have*

$$\mathcal{F} \cong \lim_{\tau \geq -n} \mathcal{F}.$$

  Then the **analytification functor** $(-)^{\mathbf{an}}$ *commutes with this limit.*

- *([PY16, Theorems 7.4 and 7.5]) Let $\mathscr{X}$ be a proper algebraic stack over* $\mathrm{Spec}(A)$. *The analytification functor on coherent sheaves induces an equivalence of categories*

$$\mathrm{Coh}(\mathscr{X}) \xrightarrow{\cong} \mathrm{Coh}(\mathscr{X}^{an}).$$

Geometry of Drinfeld Modular Forms (1/3)

Let q be odd;
Let $\Gamma \leq \mathrm{GL}_2(A)$;
Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma_2} \\ \downarrow \\ \mathscr{X}_\Gamma \end{array}$$

When we compute the log canonical ring $R(\mathscr{X}_{\Gamma_2}; 2\Delta)$ we get the following result.

Theorem ([Fra23, 6.1])

There is an isomorphism of graded rings

$$M(\Gamma_2) \cong R(\mathscr{X}_{\Gamma_2}; \Omega^1_{\mathscr{X}_{\Gamma_2}}(2\Delta)),$$

given by isomorphisms

$$M_{k,l}(\Gamma_2) \to H^0(\mathscr{X}_{\Gamma_2}, \Omega^1_{\mathscr{X}_{\Gamma_2}}(2\Delta)^{\otimes k/2})$$

of form $f \mapsto f(dz)^{\otimes k/2}$, *where* $k \equiv 2l \pmod{q-1}$.

Geometry of Drinfeld Modular Forms (2/3)

Let q be odd;
Let $\Gamma \leq \mathrm{GL}_2(A)$;
Let $\Gamma_2 = \{\gamma \in \Gamma : \det(\gamma) \in (\mathbb{F}_q^\times)^2\}$.
Consider the cover of modular curves

$$\begin{array}{c}\mathscr{X}_{\Gamma_2} \\ \big\downarrow \\ \mathscr{X}_{\Gamma}\end{array}$$

When we compare the modular forms for Γ and Γ_2 we find the following.

Theorem ([Fra23, 6.2])

We have $M(\Gamma) \cong M(\Gamma_2)$, with

$$M_{k,l}(\Gamma_2) = M_{k,l_1}(\Gamma) \oplus M_{k,l_2}(\Gamma)$$

on each component,where l_1, l_2 are the solutions to $k \equiv 2l \pmod{q-1}$.

Geometry of Drinfeld Modular Forms (3/3)

Let q be odd;
Let $\Gamma \leq \mathrm{GL}_2(A)$;
Let $\Gamma_1 = \{\gamma \in \Gamma : \det(\gamma) = 1\}$.
Suppose that $\Gamma_1 \leq \Gamma' \leq \Gamma$.
Consider the cover of modular curves

$$\begin{array}{c} \mathscr{X}_{\Gamma'} \\ \downarrow \\ \mathscr{X}_{\Gamma} \end{array}$$

When we compare the modular forms for Γ and Γ' we find the following generalization of [Fra23, Theorem 6.2].

Theorem ([Fra23, 6.12])

We have $M(\Gamma) \cong M(\Gamma')$, and each component $M_{k,l}(\Gamma')$ is some direct sum of components $M_{k,l'}(\Gamma)$ for some nontrivial l'.

Conclusion

Thank you!

Further details available at arXiv:2310.19623

and at arXiv:2312.15128

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