A New Mathematical Approach to the Problem of Impredicativity

Application to Ecological Economics * General Paradigm *Framework for Quantitative Models

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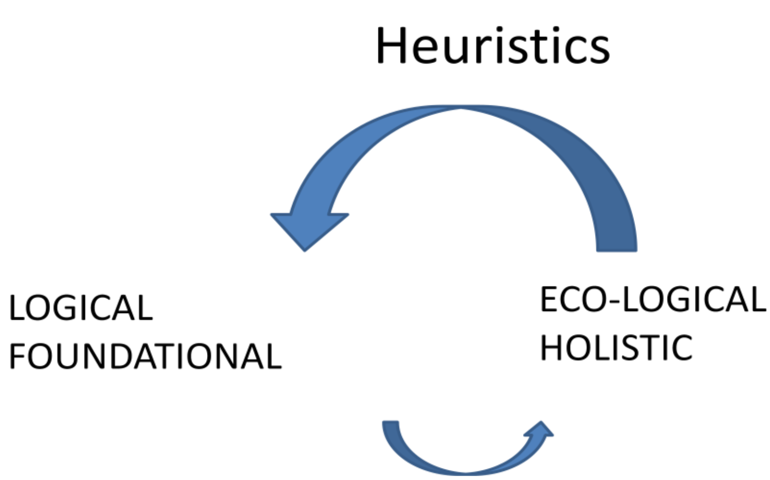
The following paper reports on some joint work with Mark Lance[1] on a foundational approach to mathematical propositions of a sufficiently complex sort and explores the connection to ecological economics. Formally these propositions are referred to as Σ^1_2 sentences (the superscript referring to objects of type 1 and the 2 referring to the number of alternations of quantifiers. The key feature of these mathematical objects is that they exhibit very explicitly the feature of having their very natures or definitions being inextricably tied to the global environment of which they form an integral part. The formal problem that results from this is commonly referred to as *impredicativity*.

More generally, the abstract setting that we find ourselves in is reflective of what we encounter in ecological contexts. Indeed the heuristics involved in understanding the naturalistic setting would seem intimately connected to what is guiding the development of new ideas in the logical mathematical setting. Indeed, although this is essentially outside the scope of this paper, it appears that these ideas are essential in overcoming fundamental foundational obstacles.

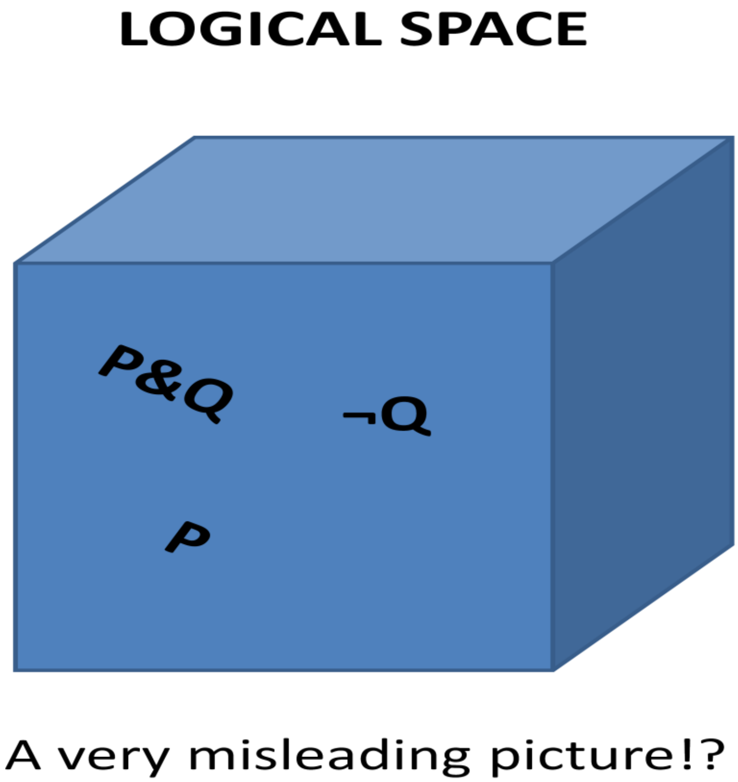
Now, one might expect that the precise articulation of these ideas in the abstract would, in turn, yield in the very least some insights about complex systems and perhaps even some formal tools for modeling such phenomena. The interaction between these two settings is illustrated in Figure 1. The upper heavy arrow indicates what has clearly proved useful and the lighter lower arrow indicated what is to be explored and hopefully amplified here. Ultimately, as the diagram suggests, we would anticipate a dialogue between the settings strengthening overall the influence in each direction. Further comments concerning this will be made at the conclusion of this paper.

FIGURE 1



In regards to the upper arrow, it might be thought that logical and foundational issues are independent and perhaps totally prior to ecological ones which are after all entirely contingent, or so we might say. Furthermore, logic might be visualized of as providing an immutable and antiseptic environment which can be given content subject only to explicit and clearly defined constraints. A picture is presented in Figure 2. While for many purposes this picture can be useful, deeper investigations into the nature of logical and mathematical foundations reveal that such a picture can be highly misleading. There is no once and for all logical foundation independent of content. When the content is specifically engineered to be simple and self contained then this may give the illusion that Figure 2 is workable, however whenever there is a potential for the content and the logical environment that contains it to interact then we are presented with puzzles and problems. Semantic paradoxes have been well known since ancient times. With the advent of modern logic it has become clear that these problems are deep and significant. In particular, attempts to give comprehensive logical foundations have been hampered by the impredicativity problem. Typically, one encounters this problem in the form of real numbers whose definition involves in an essential way the full set of real numbers.

Figure 2



Turning our attention to real world settings, Giampietro and Mayumi[2] identify *impredicativity* specifically as one of four epistemological challenges for ecological economics. Impredicative objects can be seen informally to arise quite naturally when there is phenomena that interacts with an environment or a set of objects containing the object to be defined.

In the abstract setting it is convenient to represent real numbers as functions from non-negative integers to non-negative integers. This is a flexible setting that can be used to code many common mathematical objects. For example, continuous functions on real numbers can be represented by functions from rational numbers to rational numbers where the rationals are in turn coded by integers. Alternatively such functions can be used to give values of physical or economic variables at discrete time inputs.

Formally, we define the Baire Space to be $\{f \mid f: N \rightarrow N\}$ where N denotes the non-negative integers. For example, f(0)=1, f(1)=2, f(2)=2, f(3)=3……f(n)=n…………would be an element of the Baire Space. Each such f can be identified with a "real number" through an infinite continued fraction . Although this is not essential to what follows, it can be shown that the Baire Space in its natural topology (tree) is homeomorphic to the irrationals (in their usual subspace topology).

We restrict our attention to certain elements of this space which have restrictive definitions yet are complicated enough to depend on the whole set of objects that are to be defined. The definitional form is as follows: $\{x \mid \exists Y\ \forall X\ \exists s\ \varphi_k(Y, X, s, x)\}$ where $\varphi_0, \varphi_1, \ldots \varphi_k, \ldots$ is an effective listing of all bounded formulas of 2^{nd} order arithmetic. Capital letters denote elements of the Baire Space and small case letters represent elements of N, the non-negative integers. Alternatively, we can think of these formulas as representing computation procedures (in our favorite computing language) which can also take as inputs "oracles" or other elements of the Baire Space which are in turn computed by the same procedures. Also we can regard individual elements "Y" as potentially coding a multitude of individual elements or environment. So we can think of an element x_0 as belonging to some set $\{x|\exists Y\ \forall X\ \exists s\ \varphi_k(Y, X, s, x)\}$ if there is an environment Y such that for all X there exists a convergent computation according to φ_k (represented by s) relative to X and Y.

We then investigate the construction of Y's which witness the truth of such formulas. We do this in stages. Below we first give the technical details and then give an informal explanation.

Stage $\beta = 0$: $A_{<>} = A^{\beta}_{<>} = <0,0,0,\ldots>$ for all β (choice of $A_{<>}$ is arbitrary)

Let $Q\text{^}(0) = \{ A_{<>}\}$

Stage $\beta = \alpha+1$ (successor stage):
Here we have a simple (ω-length) inductive sub-definition.

First, we let $\varphi_0, \varphi_1, \ldots \varphi_k, \ldots$ be an effective listing of all bounded 2nd order formulas (with five free variables three being second order variables). Equivalently chose an appropriate universal such formula.

At each level of our sub-definition we will produce, for each φ_k and each set previously constructed at the previous level, acting as an oracle, an approximation of the set of x's satisfying that formula. So we have this tree of definitions for the x's that satisfy φ_k relativized to previously constructed sets on the tree. Note that we identify a set of natural numbers with the element of the Baire Space that is its characteristic function.

level n+1:

Let $\rho\in\omega^n$ (note that $\omega^0 = \{<>\}$)

$A^{\beta}_{\rho\text{^}<k>} = \{x|\exists Y\in Q\text{^}(\alpha)\ \forall X\ \exists s\ \varphi_k(Y, X, s, x, A^{\beta}_{\rho})\}$

(Note that this recursively defines A^{β}_{ρ} for each $\rho\in\omega^{<\omega}$ and in particular, when n=0 we have

$A^{\beta}_{<k>} = \{x|\exists Y\in Q\text{^}(\alpha)\forall X\exists s\varphi_k(Y, X, s, x, A_{<>})\}$)

Note: the A^{α}_{ρ} are functioning to create relativized versions *of the* φ_k*'s* which are functioning to give us more extensive syntactic access to what may be lurking below the surface and some point provide us with new witnesses to sentences of this type. Also note that while and we could approximate the $\forall X$ part of the sentence we simply evaluate it in one step. It can be shown that this is not problematic in terms of impredicativity issues. In technical terms, we can show that this evaluation of the universal quantifier is fully absolute.

We are now in a position to define

$Q\text{^}(\beta) = \cup_{\alpha<\beta} \{ A^{\alpha}_{\rho}|\rho\in\omega^{<\omega}\}$

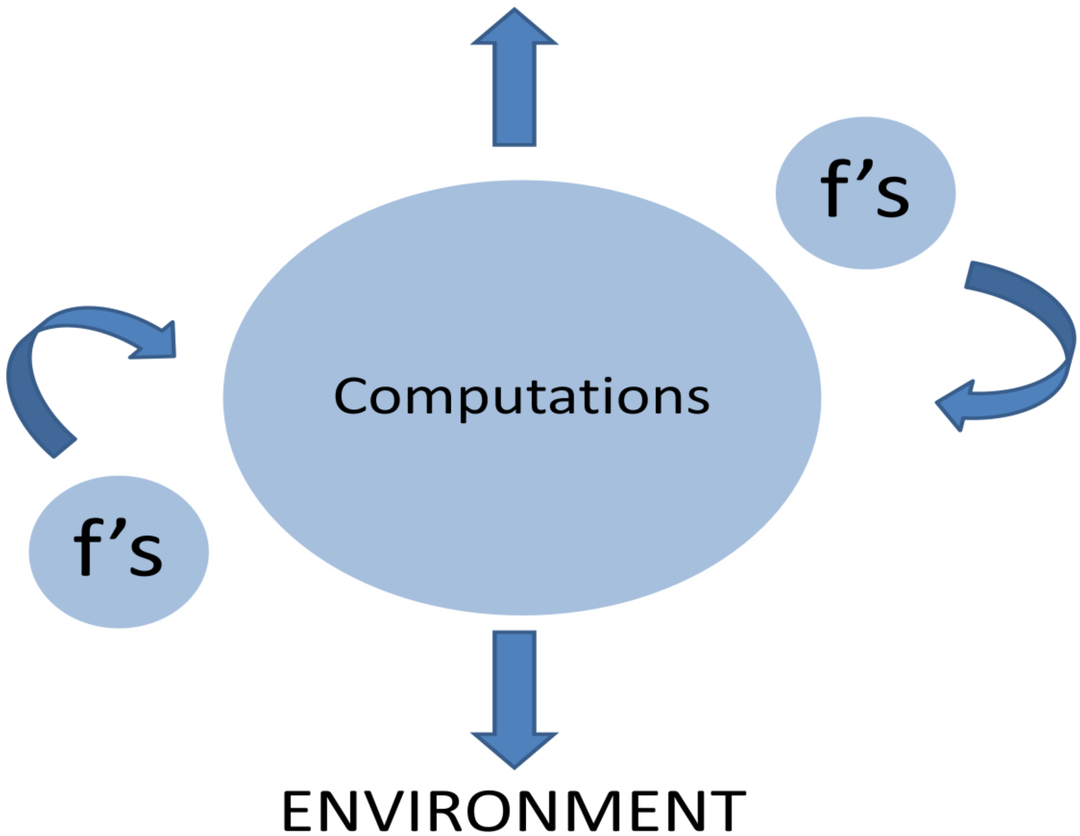
Stage β = limit

Let $A^{\beta}_{\rho} = \lim_{\alpha<\beta} A^{\alpha}_{\rho} = \{x|\exists\alpha'<\beta\ \forall\alpha(\alpha'<\alpha<\beta \rightarrow x \in A^{\alpha}_{\rho})\}$ and

$Q\text{^}(\beta) = \cup_{\alpha<\beta} \{ A^{\alpha}_{\rho}|\rho\in\omega^{<\omega}\}$.

Now each $Q\text{^}(\beta)$ is a union of sets of the form A^{β}_{ρ} . These sets can be thought of as an environment which in turn redefines at later stages the very same sets. Note that we can represent an element of the Baire Space by a set consisting of nodes which are initial finite approximations of such an element. The general picture is given in Figure 3. The large circle in the middle represents the computations that give rise to the A's defining the functions in the Baire Space represented by f's. These computations will depend at each stage on the status of the computations of these f's which in turn serves to redefine these same f's.

FIGURE 3



In summary, each stage involves an interaction of the results of the computations

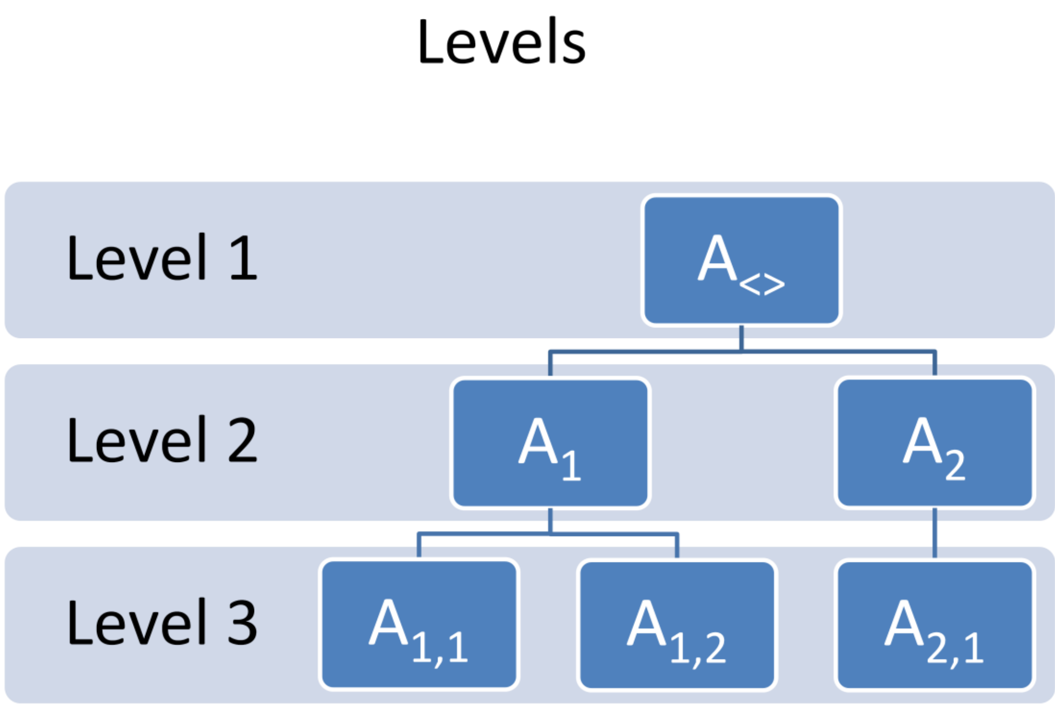
with the environment. So each stage gives a recomputation of the underlying objects making up the environment.

Approximating complicated sets stage by stage is well known in definability theory. The feature that is somewhat new here is the inclusion of embedded levels, one for each non-negative integer. This creates a natural hierarchy which is potentially redefined at every subsequent stage.

Formally, the subscripted A's represent the sets at various levels. For example, $A_{<2,1>}$ is the set of x's that satisfy φ_1 with oracle $A_{<2>}$ which in turn is the set of x's that satisfy φ_2 using as oracle $A_{<>}$, the 'null' oracle. This gives a tree of possible sets. The first few levels of such a tree is illustrated in Figure 4.

These levels may be reminiscent of the hierarchies discussed by Nicholas Georgescu-Roegen[3]. In particular we can think of the biological environment defining a social environment which then bring forth market behavior which in turn beget further derivative markets.

FIGURE 4



Now the levels only form stable hierarchies as long as there are local equilibriums, meaning that there is stability from one stage to the next.

On the other hand, long term equilibrium only obtains if The structure (or some aspect of it) is stable for all future stages

A key feature of the movement from local to global fixed points lies in a notion called saturation. Formally the following result is a key technical tool in describing how the structure described above evolves.

Saturation Lemma: If $\{T_i\}$ is (f, g) saturated then $\cap T_i \neq \varnothing$.

Pf: The proof has two steps. First, we identify a g' which lies in all the A_i's

(each represented by T_i).

To find such a g' define a (1-dim) tree T* as follows (defined by specifying the tree at level n):

$t \in T^*$ iff len(t)=n and there exist $\{s_i\}_{i<n}$ such that for all i<n, $(s_i, t) \in T_i$ and $t(i) \leq g(i)$,

and for all $j<n$, $s_i(j) \leq f(h(<i,j>))$.

T^* is closed downward and finitely branching by definition. T^* is infinite since $\{T_i\}_{i \in w}$ is (f, g) saturated.

By König's Lemma there exists a g' (which we can take as the leftmost path) lying on T^*.

Second, we define for each i, f_i such that (f_i, g') lies on T_i. Fix i. As above define a tree T_i^* (at level n) such that:

$s \in T_i^*$ iff len(s)=n and there exist $t \subset g'$ such that $(s, t) \in T_i$ and for all $j<n$, $s(j) \leq f(h(<i, j>))$.

T_i^* is closed downward and finitely branching by definition. T_i^* is infinite since for every n there exists a s such that $(s, g'|n) \in T_i$ and for all $j<n$, $s(j) \leq f(h(<i, j>)$ by the definition of T^* and the fact that g' lies on T^*. Again apply König's Lemma to obtain f_i lying on T_i^*.

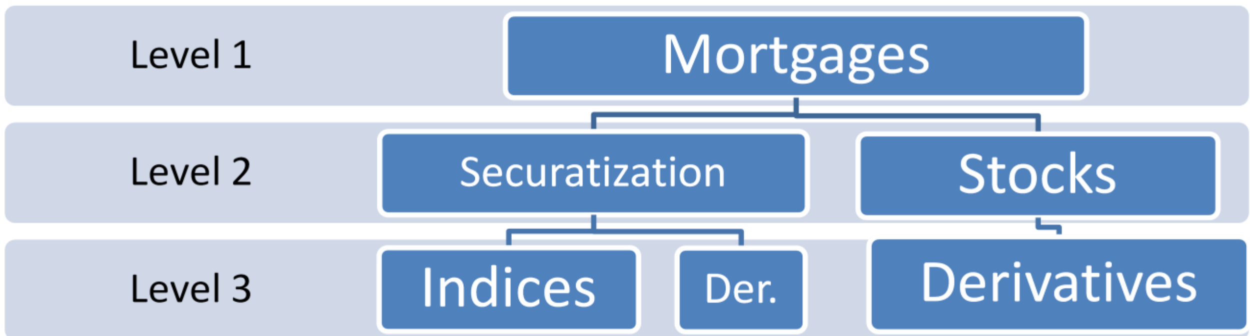
Now (f_i, g') lies on T_i, since given $s \subset f_i$ there must be a (unique) $t \subset g'$ of the same length such that $(s, t) \in T_i$ by the definition of T_i^* and the fact that f_i lies on T_i^*. Therefore $g' \in \cap T_i$.

Local equilibrium can be prescribed by a system of axioms. However, global equilibrium can only be characterized by consideration of all levels acting simultaneously.

Lastly, as an example we describe in Figure 5 the recent financial bubble as giving such a hierarchy.

Figure 5

Financial Bubble



[1] Lance and Mourad, "Semantic Self-Awareness and Determinate Truth Values in Second Order Arithmetic" Camp Out! A conference in honor of Joe Camp on the occasion of his retirement. April 7–8 2006

[2] Giampietro and Mayumi, The Evolution of Complex Adaptive Systems and the Challenge for Scientific Analysis in Polimeni, Mayumi, Giampietro and Alcot: The Jevons Paradox and the Myth of Resource Efficiency (Earthscan, London, 2008).

[3] Georgescu-Roegen: The Entropy Law and the Economic Process (Harvard University Press, Cambridge Massachusetts, 1971).