CHAPTER
TWO

THE THEORY OF CONSUMER BEHAVIOR

The postulate of rationality is the customary point of departure in the theory of the consumer's behavior. The consumer is assumed to choose among the available alternatives in such a manner that the satisfaction derived from consuming commodities (in the broadest sense) is as large as possible. This implies that he is aware of the alternatives facing him and is capable of evaluating them. All information pertaining to the satisfaction that the consumer derives from various quantities of commodities is contained in his *utility function.*

The concepts of utility and its maximization are void of any sensuous connotation. The assertion that a consumer derives more satisfaction or utility from an automobile than from a suit of clothes means that if he were presented with the alternatives of receiving as a gift either an automobile or a suit of clothes, he would choose the former. Things that are necessary for survival—such as vaccine when a smallpox epidemic threatens—may give the consumer the most utility, although the act of consuming such a commodity has no pleasurable sensations connected with it.

The nineteenth-century economists W. Stanley Jevons, Léon Walras, and Alfred Marshall considered utility measurable, just as the weight of objects is measurable. The consumer was assumed to possess a *cardinal* measure of utility, i.e., he was assumed to be capable of assigning to every commodity or combination of commodities a number representing the amount or degree of utility associated with it. The numbers representing amounts of utility could be manipulated in the same fashion as weights. Assume, for example, that the utility of A is 15 units and the utility of B 45 units. The consumer would "like" B three times as strongly as A. The differences between utility numbers could be compared, and the comparison could lead to a statement such as "A is preferred to B twice as much as C is preferred to D." It was also assumed by the nineteenth-century economists that the additions to a consumer's total utility resulting from consuming additional units of a commodity decrease as he consumes more of it. The consumer's behavior can be deduced from the above assumptions. Imagine that a certain price, say 2 dollars, is charged for coconuts. The consumer, confronted with coconuts, will not buy any if the amount of utility he surrenders by paying the price of a coconut (i.e., by parting with purchasing power) is greater than the utility he gains by consuming it. Assume that the utility of a dollar is 5 utils and remains approximately constant for small variations in income, and that the consumer derives the following increments of utility by consuming an additional coconut:

Unit	Additional utility
Coconut 1	20
Coconut 2	9
Coconut 3	7

The consumer will buy at least one coconut, because he surrenders 10 utils in exchange for 20 utils and thus increases his total utility.[1] He will not buy a second coconut, because the utility loss exceeds the gain. In general, the consumer will not add to his consumption of a commodity if an additional unit involves a net utility loss. He will increase his consumption only if he realizes a net gain of utility from it. For example, assume that the price of coconuts falls to 1.6 dollars. Two coconuts will now be bought. A fall in the price has increased the quantity bought. This is the sense in which the theory predicts the consumer's behavior.

The assumptions on which the theory of cardinal utility is built are very restrictive. Equivalent conclusions can be deduced from much weaker assumptions. Therefore it will *not* be assumed in the remainder of this chapter that the consumer possesses a cardinal measure of utility or that the additional utility derived from increasing his consumption of a commodity diminishes.

If the consumer derives more utility from alternative A than from alternative B, he is said to prefer A to B.[2] The postulate of rationality is equivalent to the following statements: (1) for all possible pairs of alternatives A and B the consumer knows whether he prefers A to B or B to A, or

[1] The price is 2 dollars; the consumer loses 5 utils per dollar surrendered. Therefore the gross loss is 10 utils, and the gross gain is 20 utils.

[2] A chain of definitions must eventually come to an end. The word "prefer" can be defined to mean "would rather have than," but then this expression must be left undefined. The term "prefer" is also void of any connotation of sensuous pleasure.

whether he is indifferent between them; (2) only one of the three possibilities is true for any pair of alternatives; (3) if the consumer prefers A to B and B to C, he will prefer A to C. The last statement ensures that the consumer's preferences are consistent or *transitive*: if he prefers an automobile to a suit of clothes and a suit of clothes to a bowl of soup, he must prefer an automobile to a bowl of soup.

The postulate of rationality, as stated above, merely requires that the consumer be able to rank commodities in order of preference. The consumer possesses an *ordinal* utility measure; i.e., he need not be able to assign numbers that represent (in arbitrary units) the degree or amount of utility that he derives from commodities. His ranking of commodities is expressed mathematically by his utility function. It associates certain numbers with various quantities of commodities consumed, but these numbers provide only a ranking or ordering of preferences. If the utility of alternative A is 15 and the utility of B is 45 (i.e., if the utility function associates the number 15 with alternative or commodity A and the number 45 with alternative B), one can only say that B is preferred to A, but it is meaningless to say that B is liked three times as strongly as A. This reformulation of the postulates of the theory of consumer behavior was effected only around the turn of the last century. It is remarkable that the consumer's behavior can be explained just as well in terms of an ordinal utility function as in terms of a cardinal one. Intuitively one can see that the consumer's choices are completely determinate if he possesses a ranking (and only a ranking) of commodity bundles according to his preferences. One could visualize the consumer as possessing a rank-ordered list of all conceivable alternative commodity bundles that can be purchased for a sum of money equal to his income; when the consumer receives his income, he simply purchases the bundle at the top of the list.[1] Therefore it is not necessary to assume that he possesses a cardinal measure of utility. The much weaker assumption that he possesses a consistent ranking of preferences is sufficient.

The basic tools of analysis and the nature of the utility function are discussed in Sec. 2-1. The individual consumer's optimum consumption levels are determined in Sec. 2-2, and it is shown that the solution of the consumer's maximum problem is invariant with respect to positive monotonic transformations of his utility function. Demand functions are derived in Sec. 2-3, and the analysis is extended to the problem of choice between income and leisure in Sec. 2-4. The effect of price and income variations on consumption levels is examined in Sec. 2-5. The theory is finally generalized to an arbitrary number of commodities in Sec. 2-6.

[1] How much a particular bundle on the list is liked is irrelevant; only the bundle at the top of the list will be purchased.

2-1 BASIC CONCEPTS

The Nature of the Utility Function

Consider the simple case in which the consumer's purchases are limited to two commodities. His ordinal utility function is

$$U = f(q_1, q_2) \tag{2-1}$$

where q_1 and q_2 are the quantities of the two commodities Q_1 and Q_2 which he consumes. It is assumed that $f(q_1, q_2)$ is continuous, has continuous first- and second-order partial derivatives, and is a regular strictly quasi-concave function.[1] Furthermore, it is assumed that the partial derivatives of (2-1) are strictly positive. This means that the consumer will always desire more of both commodities. These assumptions are sometimes modified to cover special cases. Nonnegative consumption levels normally constitute the domain for the utility function, though in some cases the domain is limited to positive levels.

The consumer's utility function is not unique (see Sec. 2-2). In general, any single-valued increasing function of q_1 and q_2 can serve as a utility function. The utility number U^0 assigned to any particular commodity combination indicates that it is preferable or superior to all combinations with lower numbers and inferior to those with higher numbers.

The utility function is defined with reference to consumption during a specified period of time. The level of satisfaction that the consumer derives from a particular commodity combination depends upon the length of the period during which he consumes it. Different levels of satisfaction are derived from consuming ten portions of ice cream within one hour and within one month. There is no unique time period for which the utility function *should* be defined. However, there are restrictions upon the possible length of the period. The consumer usually derives utility from variety in his diet and diversification among the commodities he consumes. Therefore, the utility function must not be defined for a period so short that the desire for variety cannot be satisfied. On the other hand, tastes (the shape of the function) may change if it is defined for too long a period. Any intermediate period is satisfactory for the static theory of consumer behavior.[2] The present theory is static in the sense that the utility function is defined with reference to a single time period, and the consumer's optimal expenditure pattern is analyzed only

[1] A strictly quasi-concave function for which $2f_{12}f_1f_2 - f_{11}f_2^2 - f_{22}f_1^2 > 0$. The positiveness of the expression makes certain derivations easier, and the assumption of regularity is made for convenience, since strict quasi-concavity alone guarantees only the weak inequality $\geqq$. See Sec. A-3.

[2] The theory would break down if it were impossible to define a period that is neither too short from the first point of view nor too long from the second.

with respect to this period. No account is taken of the possibility of transferring consumption expenditures from one period to another.[1]

Indifference Curves

A particular level of utility or satisfaction can be derived from many different combinations of Q_1 and Q_2.† For a given level of utility U^0, Eq. (2-1) becomes

$$U^0 = f(q_1, q_2) \tag{2-2}$$

where U^0 is a constant. Since the utility function is continuous, (2-2) is satisfied by an infinite number of combinations of Q_1 and Q_2. Imagine that the consumer derives a given level of satisfaction U^0 from 5 units of Q_1 and 3 units of Q_2. If his consumption of Q_1 were decreased from 5 to 4 without an increase in his consumption of Q_2, his satisfaction would certainly decrease. In general, it is possible to compensate him for the loss of 1 unit of Q_1 by allowing an increase in his consumption of Q_2. Imagine that an increase of 3 units in his consumption of Q_2 makes him indifferent between the two alternative combinations. Other commodity combinations which yield the consumer the same level of satisfaction can be discovered in a similar manner. The locus of all commodity combinations from which the consumer derives the same level of satisfaction forms an *indifference curve.* An *indifference map* is a collection of indifference curves corresponding to different levels of satisfaction. The quantities q_1 and q_2 are measured along the axes of Fig. 2-1. One indifference curve passes through every point in the positive quadrant of the q_1q_2 plane. Indifference curves correspond to higher and higher levels of satisfaction as one moves in a northeasterly direction in Fig. 2-1. A movement from point A to point B would increase the consumption of both Q_1 and Q_2. Therefore B must correspond to a higher level of satisfaction than A.

Indifference curves cannot intersect as shown in Fig. 2-2. Consider the points A_1, A_2, and A_3. Let the consumer derive the satisfaction U_1 from the batch of commodities represented by A_1 and similarly U_2 and U_3 from A_2 and A_3. The consumer has more of both commodities at A_3 than at A_1, and therefore $U_3 > U_1$. Since A_1 and A_2 are on the same indifference curve,

[1] The present analysis is static in that it does not consider what happens after the current income period. The consumer makes his calculations for only one such period at a time. At the end of the period he repeats his calculations for the next one. If the consumer were capable of borrowing, one would consider his total liquid resources available in any time period instead of his income proper. Conversely, he may save, i.e., not spend all his income on consumption goods. Provision can be made for both possibilities without changing the essential points of the analysis (see Sec. 12-2).

† By definition, a commodity is an item of which the consumer would rather have more than less. Otherwise he is dealing with a discommodity. In reality a commodity may become a discommodity if its quantity is sufficiently large. For example, if the consumer partakes of too many portions of ice cream, it may become a discommodity for him. It is assumed in the remainder of the chapter that such a point of saturation has not been reached.

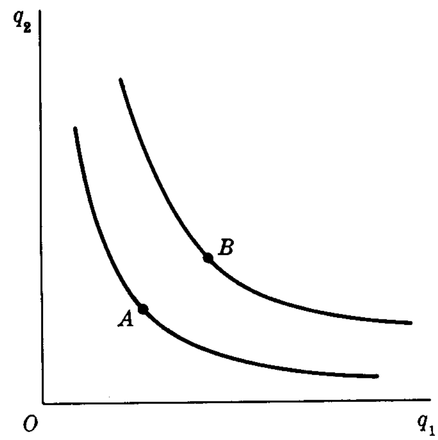

Figure 2-1

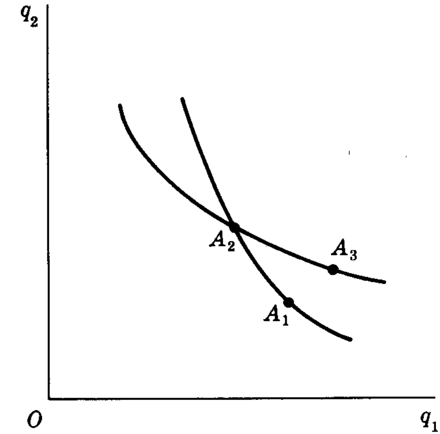

Figure 2-2

$U_1 = U_2$. The points A_2 and A_3 are also on the same indifference curve, and therefore $U_2 = U_3$. This implies $U_1 = U_3$. Therefore, A_1 and A_3 are on the same indifference curve, contrary to assumption.

The assumption that the utility function is strictly quasi-concave restricts the shape of the indifference curves. Consider two distinct points on a given indifference curve where $U^0 = f(q_1^0, q_2^0) = f(q_1^{(1)}, q_2^{(1)})$. Strict quasi-concavity (see Sec. A-2) ensures that

$$U[\lambda q_1^0 + (1-\lambda) q_1^{(1)}, \lambda q_2^0 + (1-\lambda) q_2^{(1)}] > U^0$$

for all $0 < \lambda < 1$. Thus, all interior points on a line segment connecting two points on an indifference curve lie on higher indifference curves. This means that an indifference curve expresses q_2 as a strictly convex function of q_1, sometimes expressed by saying that indifference curves are "convex to the origin."

The Rate of Commodity Substitution

The total differential of the utility function is

$$dU = f_1\, dq_1 + f_2\, dq_2 \tag{2-3}$$

where f_1 and f_2 are the partial derivatives of U with respect to q_1 and q_2. The total change in utility (compared to an initial situation) caused by variations in q_1 and q_2 is approximately the change in q_1 multiplied by the change in utility resulting from a unit change of q_1 plus the change in q_2 multiplied by the change in utility resulting from a unit change in q_2. Let the consumer move along one of his indifference curves by giving up some Q_1 in exchange for Q_2. If his consumption of Q_1 decreases by dq_1 (therefore, $dq_1 < 0$), the resulting loss of utility is approximately $f_1\, dq_1$. The gain of utility caused by acquiring some Q_2 is approximately $f_2\,dq_2$ for similar reasons. Taking arbitrarily small increments, the sum of these two terms must equal zero in the limit, since the total change in utility along an indifference curve is zero by definition.[1] Since the analysis runs in terms of ordinal utility functions, the magnitudes of $f_1\,dq_1$ and $f_2\,dq_2$ are not known. However, it must still be true that the sum of these terms is zero. Setting $dU = 0$,

$$f_1\,dq_1 + f_2\,dq_2 = 0$$

yields

$$-\frac{dq_2}{dq_1} = \frac{f_1}{f_2} \tag{2-4}$$

The slope of an indifference curve, dq_2/dq_1, is the rate at which a consumer would be willing to substitute Q_1 for Q_2 per unit of Q_1 in order to maintain a given level of utility. The negative of the slope, $-dq_2/dq_1$, is the *rate of commodity substitution* (RCS) of Q_1 for Q_2, and it equals the ratio of the partial derivatives of the utility function.[2] The reciprocal of the RCS is the rate at which the consumer would be willing to substitute Q_2 for Q_1 per unit of Q_2.

In a cardinal analysis the partial derivatives f_1 and f_2 are defined as the *marginal utilities* of the commodities Q_1 and Q_2.† This definition is retained in the present ordinal analysis. However, the partial derivative of an ordinal utility function cannot be given a cardinal interpretation. Therefore, the numerical magnitudes of individual marginal utilities are without meaning. The consumer is not assumed to be aware of the existence of marginal utilities, and only the economist need know that the consumer's RCS equals the ratio of marginal utilities. The signs as well as the ratios of marginal utilities are meaningful in an ordinal analysis. A positive value for f_1 signifies that an increase in q_1 will increase the consumer's satisfaction level and move him to a higher indifference curve.

Since the utility function is a regular strictly quasi-concave function, the strict inequality

$$2f_{12}f_1f_2 - f_{11}f_2^2 - f_{22}f_1^2 > 0 \tag{2-5}$$

is satisfied at each point within its domain (see Sec. A-3). By further differentiation of (2-4) the rate of change of the slope of the indifference curve is[3]

$$\frac{d^2q_2}{dq_1^2} = -\frac{1}{f_2^3}(f_{11}f_2^2 - 2f_{12}f_1f_2 + f_{22}f_1^2) \tag{2-6}$$

[1] Imagine the utility function as a surface in three-dimensional space. Then the total differential (2-3) is the equation of the tangent plane to this surface at some point. This justifies the use of the word "approximate" in the above argument (see Sec. A-2).

[2] The rate of commodity substitution is frequently referred to in the literature of economics as the *marginal* rate of substitution. Cf. J. R. Hicks, *Value and Capital* (2d ed., Oxford: Clarendon Press, 1946), pt. I.

† The marginal utility of a commodity is often loosely defined as the increase in utility resulting from a unit increase in its consumption.

[3] Note that (2-6) is obtained by taking the total derivative of the slope of the indifference curve instead of the partial derivative.

Inequality (2-5) ensures that the parenthesized term on the right-hand side of (2-6) is negative. Since $f_2 > 0$, regular strict quasi-concavity dictates that the negative slope of the indifference curve becomes larger algebraically and smaller in absolute value as Q_1 is substituted for Q_2. The indifference curve becomes flatter, and the RCS, which is the absolute value of its slope, decreases. As the consumer moves along an indifference curve, he acquires more Q_1 and less Q_2, and the rate at which he is willing to sacrifice Q_2 to acquire yet more Q_1 declines. The increasing relative scarcity of Q_2 increases its relative value to the consumer, and the increasing relative abundance of Q_1 decreases its relative value.

Existence of the Utility Function

It is not intuitively obvious that real-valued functions that can serve as utility functions exist for all consumers. A consumer's preferences must satisfy certain conditions in order to be representable by a utility function. A set of sufficient conditions for the existence of a utility function is expressed in the following assumptions:

1. The various commodity combinations available to the consumer stand in a relation to each other, denoted by R. The meaning of R is "is at least as well liked as." The relation R is complete: For any pair of commodity combinations $\mathbf{A}_1$ and $\mathbf{A}_2$ either $\mathbf{A}_1 R \mathbf{A}_2$, $\mathbf{A}_2 R \mathbf{A}_1$, or both. Furthermore, R is reflexive: $\mathbf{A}_1 R \mathbf{A}_1$, whatever $\mathbf{A}_1$ may be. Finally, R is transitive: If $\mathbf{A}_1 R \mathbf{A}_2$ and $\mathbf{A}_2 R \mathbf{A}_3$, then $\mathbf{A}_1 R \mathbf{A}_3$.
2. The set of all commodity combinations available to the consumer is connected. If $\mathbf{A}_1$ and $\mathbf{A}_2$ are available to the consumer, one can find a continuous path of available combinations connecting $\mathbf{A}_1$ and $\mathbf{A}_2$.
3. Given some commodity combination $\mathbf{A}_1$, one may consider the set of all combinations at least as well liked as $\mathbf{A}_1$ and the set of all combinations not more liked than $\mathbf{A}_1$. These two sets are closed. This means that if one selected for consideration an infinite sequence of commodity combinations which converged to some limiting combination $\mathbf{A}_0$, and if each member of the sequence were at least as well liked as $\mathbf{A}_1$, then the limiting combination would also be at least as well liked as $\mathbf{A}_1$. This condition ensures the continuity of the consumer's preferences and rules out "jumps." It ensures, for example, that if two commodity combinations differ from each other only slightly and if one of these is preferred to some given combination $\mathbf{A}_1$, then the other will be at least as well liked as $\mathbf{A}_1$.

It might seem that these conditions are so unrestrictive as to be almost always satisfied. It is easy, however, to cite preference structures that do not satisfy them. Consider the following case. Let there be two commodities Q_1 and Q_2, and consider two commodity combinations $\mathbf{A}_1 = (q_1^{(1)}, q_2^{(1)})$ and $\mathbf{A}_2 = (q_1^{(2)}, q_2^{(2)})$. Imagine that the preference structure of the consumer is given by the following rule: $\mathbf{A}_1$ is preferred to $\mathbf{A}_2$ if either $q_1^{(1)} > q_1^{(2)}$ or $q_1^{(1)} = q_1^{(2)}$ and $q_2^{(1)} > q_2^{(2)}$. In this situation the preference ordering is said to be *lexicographic* and no utility function exists.

The lexicographic ordering violates the third of the above assumptions. Consider the combination $\mathbf{A} = (q_1^0, q_2^0)$, and let Δq_1 and Δq_2 denote positive increments from $\mathbf{A}$. The combination $(q_1^0 + \Delta q_1, q_2^0 - \Delta q_2)$ is preferred to $\mathbf{A}$ by virtue of the lexicographic ordering. Select particular positive values for the increments, and consider the infinite sequence of commodity combinations the ith member of which is

$$\mathbf{A}_i = (q_1^0 + (\tfrac{1}{2})^i \Delta q_1, q_2^0 - \Delta q_2)$$

$\mathbf{A}_i$ is clearly preferred to $\mathbf{A}$ for any i, but the limit of the sequence

$$\lim_{i \to \infty} \mathbf{A}_i = (q_1^0, q_2^0 - \Delta q_2)$$

is inferior to $\mathbf{A}$ in violation of the third assumption.

2-2 THE MAXIMIZATION OF UTILITY

The rational consumer desires to purchase a combination of Q_1 and Q_2 from which he derives the highest level of satisfaction. His problem is one of maximization. However, his income is limited, and he is not able to purchase unlimited amounts of the commodities. The consumer's budget constraint can be written as

$$y^0 = p_1 q_1 + p_2 q_2 \tag{2-7}$$

where y^0 is his (fixed) income and p_1 and p_2 are the prices of Q_1 and Q_2 respectively. The amount he spends on the first commodity ($p_1 q_1$) plus the amount he spends on the second ($p_2 q_2$) equals his income (y^0).

The First- and Second-Order Conditions

The consumer desires to maximize (2-1) subject to (2-7). Form the Lagrange function

$$V = f(q_1, q_2) + \lambda (y^0 - p_1 q_1 - p_2 q_2) \tag{2-8}$$

where λ is an as yet undetermined multiplier (see Sec. A-3). The first-order conditions are obtained by setting the first partial derivatives of (2-8) with respect to q_1, q_2, and λ equal to zero:

$$\begin{aligned} \frac{\partial V}{\partial q_1} &= f_1 - \lambda p_1 = 0 \\ \frac{\partial V}{\partial q_2} &= f_2 - \lambda p_2 = 0 \\ \frac{\partial V}{\partial \lambda} &= y^0 - p_1 q_1 - p_2 q_2 = 0 \end{aligned} \tag{2-9}$$

Transposing the second terms in the first two equations of (2-9) to the right and dividing the first by the second yields

$$\frac{f_1}{f_2} = \frac{p_1}{p_2} \tag{2-10}$$

The ratio of the marginal utilities must equal the ratio of prices for a maximum. Since f_1/f_2 is the RCS, the first-order condition for a maximum is expressed by the equality of the RCS and the price ratio.

The first two equations of (2-9) may also be written as

$$\frac{f_1}{p_1} = \frac{f_2}{p_2} = \lambda \tag{2-11}$$

Marginal utility divided by price must be the same for all commodities. This ratio gives the rate at which satisfaction would increase if an additional dollar were spent on a particular commodity. If more satisfaction could be gained by spending an additional dollar on Q_1 rather than Q_2, the consumer would not be maximizing utility. He could increase his satisfaction by shifting some of his expenditure from Q_2 to Q_1.

The Lagrange multiplier λ can be interpreted as the marginal utility of income. Since the marginal utilities of commodities are assumed to be positive, the marginal utility of income is positive.

The second-order condition as well as the first-order condition must be satisfied to ensure that a maximum is actually reached. Denoting the second direct partial derivatives of the utility function by f_{11} and f_{22} and the second cross partial derivatives by f_{12} and f_{21}, the second-order condition for a constrained maximum requires that the relevant bordered Hessian determinant be positive:

$$\begin{vmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} > 0 \tag{2-12}$$

Expanding (2-12)

$$2f_{12}p_1p_2 - f_{11}p_2^2 - f_{22}p_1^2 > 0 \tag{2-13}$$

Substituting $p_1 = f_1/\lambda$ and $p_2 = f_2/\lambda$ from (2-9) and multiplying through by $\lambda^2 > 0$

$$2f_{12}f_1f_2 - f_{11}f_2^2 - f_{22}f_1^2 > 0 \tag{2-14}$$

Inequality (2-14), which is the same as (2-5), is satisfied by the assumption of regular strict quasi-concavity. This assumption ensures that the second-order condition is satisfied at any point at which the first-order condition is satisfied. Inequality (2-14) is also the condition for the global univalence of solutions for Eqs. (2-9) (see Sec. A-2). Thus, regular strict quasi-concavity also ensures that constrained-utility-maximization solutions are unique.

Assume that the utility function is $U = q_1q_2$, that $p_1 = 2$ dollars, $p_2 = 5$ dollars, and that the consumer's income for the period is 100 dollars. The budget constraint is

$$100 - 2q_1 - 5q_2 = 0$$

Form the function

$$V = q_1q_2 + \lambda(100 - 2q_1 - 5q_2)$$

and set its partial derivatives equal to zero

$$q_2 - 2\lambda = 0$$

$$q_1 - 5\lambda = 0$$

$$100 - 2q_1 - 5q_2 = 0$$

Solving the three linear equations gives $q_1 = 25$, $q_2 = 10$, and $\lambda = 5$. The second-order condition holds, as the reader may verify by performing the necessary differentiation. The consumer maximizes utility by consuming this combination.

Figure 2-3 contains a graphic presentation of this example. The price line AB is the geometric counterpart of the budget constraint and shows all possible combinations of Q_1 and Q_2 that the consumer *can* purchase. Its equation is $100 - 2q_1 - 5q_2 = 0$. The consumer can purchase 50 units of Q_1 if he buys no Q_2, 20 units of Q_2 if he buys no Q_1, etc. A different price line corresponds to each possible level of income; if the consumer's income were 60 dollars, the relevant price line would be CD. The indifference curves in this example are a family of rectangular hyperbolas.[1] The consumer desires to

[1] Hyperbolas the asymptotes of which coincide with the coordinate axes.

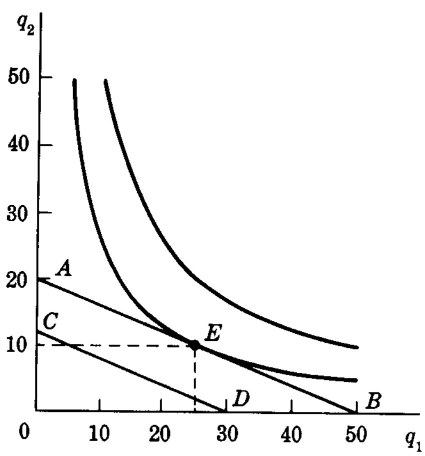

Figure 2-3

reach the highest indifference curve that has at least one point in common with AB. His equilibrium is at point E, at which AB is tangent to an indifference curve. Movements in either direction from point E result in a diminished level of utility. The constant slope of the price line, $-p_1/p_2$ or $-\frac{2}{5}$ in the present example, must equal the slope of the indifference curve. Forming the ratio of the partial derivatives of the utility function, the slope of the indifference curves in the present example is $-q_2/q_1$, and hence the RCS equals $q_2/q_1 = \frac{10}{25}$, which equals the ratio of prices as required. The second-order condition is satisfied. The indifference curves are convex, and the RCS is decreasing at the equilibrium point: $-d^2q_2/dq_1^2 = -2q_2/q_1^2 < 0$.

The Choice of a Utility Index

The numbers which the utility function assigns to the alternative commodity combinations need not have cardinal significance; they need only serve as an *index* of the consumer's satisfaction. Imagine that one wishes to compare the satisfaction a consumer derives from one hat and two shirts and from two hats and five shirts. The consumer is known to prefer the latter to the former combination. The numbers that are assigned to these combinations for the purpose of showing the strength of his preferences are arbitrary in the sense that the difference between them has no meaning. Since the second batch is preferred to the first batch, the number 3 could be assigned to the first and the number 4 to the second. However, any other set of numbers would serve as well, as long as the number assigned to the second batch exceeded that assigned to the first. Thus 3 for the first batch and 400 for the second would provide an equally satisfactory utility index. If a particular set of numbers associated with various combinations of Q_1 and Q_2 is a utility index, any positive monotonic transformation of it is also a utility index.[1] Assume that the original utility function is $U = f(q_1, q_2)$. Now form a new utility index $W = F(U) = F[f(q_1, q_2)]$ by applying a positive monotonic transformation to the original utility index. The function $F(U)$ is an increasing function of U.† It can be demonstrated that maximizing W subject to the budget constraint is equivalent to maximizing U subject to the budget constraint.

Imagine that (q_1^0, q_2^0) is the commodity bundle that uniquely maximizes $f(q_1, q_2)$ subject to the budget constraint. Let $(q_1^{(1)}, q_2^{(1)})$ be any other bundle also satisfying the budget constraint. Then by assumption $f(q_1^0, q_2^0) > f(q_1^{(1)}, q_2^{(1)})$ for any choice of $(q_1^{(1)}, q_2^{(1)})$. But by the definition of monotonicity $W(q_1^0, q_2^0) = F[f(q_1^0, q_2^0)] > F[f(q_1^{(1)}, q_2^{(1)})] = W(q_1^{(1)}, q_2^{(1)})$, which proves that the utility function $W(q_1, q_2)$ is maximized by the commodity bundle (q_1^0, q_2^0).

[1] A function $F(U)$ is a positive monotonic transformation of U if $F(U_1) > F(U_0)$ whenever $U_1 > U_0$.

† Examples are provided by the transformations $W = aU + b$, provided that a is positive, and by $W = U^2$, provided that all utility numbers are nonnegative.

Two Special Cases

The first-order conditions (2-9) are not always necessary for a maximum. Two exceptions are pictured in Fig. 2-4. In the first case (see Fig. 2-4*a*) the indifference curves are concave rather than convex; i.e., the assumption that the utility function is quasi-concave is violated. The indifference curves are bowed away from the origin, and the RCS is increasing throughout. The first-order condition for a maximum is satisfied at the point of tangency between the price line and an indifference curve, but the second-order condition is not. Therefore this point represents a local utility minimum, and the consumer can increase his utility by moving from the point of tangency toward either axis. He consumes only one commodity at the optimum. If he spends all his income on one commodity, he can buy y^0/p_1 units of Q_1 or y^0/p_2 units of Q_2. Therefore he will buy only Q_1 or only Q_2, depending upon whether $f(y^0/p_1, 0) \gtrless f(0, y^0/p_2)$. In the example shown in Fig. 2-4*a* he will buy only Q_2. In the second case (see Fig. 2-4*b*) the indifference curves have the appropriate shape, but they are everywhere less steep than the price line. Tangency is not possible; the first-order condition cannot be fulfilled because of the restrictions $q_1 \geqq 0$, $q_2 \geqq 0$. The consumer's optimum position is again given by a corner solution, and he purchases only Q_2 at the optimum.

Assume that the utility function of Fig. 2-4*b* is either strictly concave or has a positive monotonic transformation that is. The Kuhn-Tucker conditions (see Sec. A-3) are applicable. The consumer desires to maximize utility subject to the following inequality constraints:

$$y^0 - p_1q_1 - p_2q_2 \geqq 0 \qquad q_1 \geqq 0 \qquad q_2 \geqq 0$$

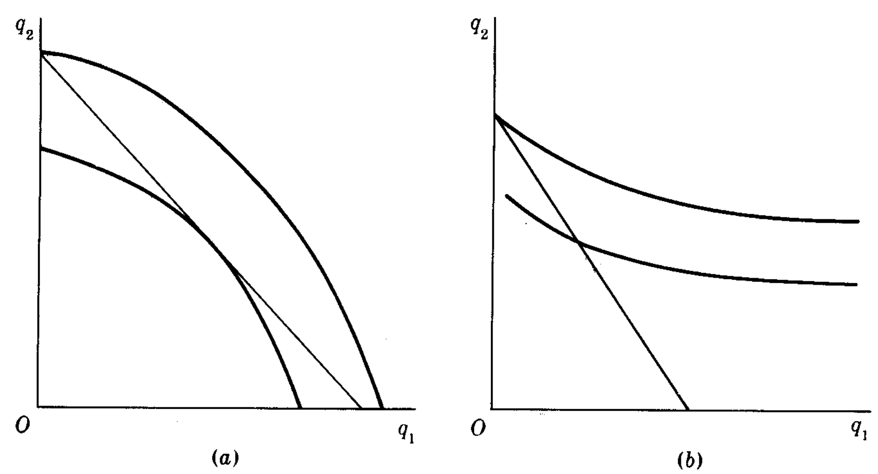

Figure 2-4

The consumer is allowed to spend less than his income, and the nonnegative consumption requirements are made explicit. The Lagrange function is again given by (2-8). The Kuhn-Tucker conditions[1] for a utility maximum require that

$$V_1 = f_1 - \lambda p_1 \leqq 0 \qquad q_1 V_1 = 0$$

$$V_2 = f_2 - \lambda p_2 \leqq 0 \qquad q_2 V_2 = 0$$

$$V_3 = y^0 - p_1 q_1 - p_2 q_2 \geqq 0 \qquad \lambda V_3 = 0$$

where the V_i are the partial derivatives of (2-8). If $f_1 > \lambda p_1$, the consumer could increase utility by increasing q_1. If $f_1 < \lambda p_1$, he could increase utility by decreasing q_1 unless q_1 already equaled zero. In the example shown in Fig. 2-4*b*, $V_1 < 0$ and $V_2 = 0$. Hence, $f_1/f_2 < p_1/p_2$.

The last pair of the Kuhn-Tucker conditions state that λ, the marginal utility of income, would equal zero if the consumer were to spend less than his income in equilibrium. However, this will not happen as long as positive marginal utilities are assumed.

2-3 DEMAND FUNCTIONS

Ordinary Demand Functions

A consumer's ordinary demand function (sometimes called a Marshallian demand function) gives the quantity of a commodity that he will buy as a function of commodity prices and his income. Ordinary demand functions are called simply demand functions unless it is necessary to distinguish them from another type of demand function. They can be derived from the analysis of utility maximization. The first-order conditions for maximization (2-9) consist of three equations in the three unknowns: q_1, q_2, and λ.† The demand functions are obtained by solving this system for the unknowns. The solutions for q_1 and q_2 are in terms of the parameters p_1, p_2, and y^0. The quantity of Q_1 (or Q_2) that the consumer purchases in the general case depends upon the prices of all commodities and his income.

As above, assume that the utility function is $U = q_1 q_2$ and the budget constraint $y^0 - p_1 q_1 - p_2 q_2 = 0$. Form the expression

$$V = q_1 q_2 + \lambda(y^0 - p_1 q_1 - p_2 q_2)$$

and set its partial derivatives equal to zero:

$$\frac{\partial V}{\partial q_1} = q_2 - p_1 \lambda = 0$$

[1] The Kuhn-Tucker conditions are necessary and sufficient given concave functions if a constraint qualification is satisfied (see Sec. A-3). This constraint qualification is assumed to be satisfied for all examples used in the text.

† Assume that the second-order conditions are fulfilled.

$$\frac{\partial V}{\partial q_2} = q_1 - p_2\lambda = 0$$

$$\frac{\partial V}{\partial \lambda} = y^0 - p_1q_1 - p_2q_2 = 0$$

Solving for q_1 and q_2 gives the demand functions:[1]

$$q_1 = \frac{y^0}{2p_1} \qquad q_2 = \frac{y^0}{2p_2}$$

The demand functions derived in this fashion are contingent on continued optimizing behavior by the consumer. Given the consumer's income and prices of commodities, the quantities demanded by him can be determined from his demand functions. Of course, these quantities are the same as those obtained directly from the utility function. Substituting $y^0 = 100$, $p_1 = 2$, $p_2 = 5$ in the demand functions gives $q_1 = 25$ and $q_2 = 10$, as in Sec. 2-2.

Two important properties of demand functions can be deduced: (1) the demand for any commodity is a single-valued function of prices and income, and (2) demand functions are homogeneous of degree zero in prices and income; i.e., if all prices and income change in the same proportion, the quantities demanded remain unchanged.

The first property follows from the strict quasi-concavity of the utility function; a single maximum, and therefore a single commodity combination, corresponds to a given set of prices and income.[2] To prove the second property assume that all prices and income change in the same proportion. The budget constraint becomes

$$ky^0 - kp_1q_1 - kp_2q_2 = 0$$

where k is the factor of proportionality. Expression (2-8) becomes

$$V = f(q_1, q_2) + \lambda(ky^0 - kp_1q_1 - kp_2q_2)$$

and the first-order conditions are

$$\begin{aligned} f_1 - \lambda k p_1 &= 0 \\ f_2 - \lambda k p_2 &= 0 \\ ky^0 - kp_1q_1 - kp_2q_2 &= 0 \end{aligned} \qquad (2\text{-}15)$$

The last equation of (2-15) is the partial derivative of V with respect to the Lagrange multiplier and can be written as

$$k(y^0 - p_1q_1 - p_2q_2) = 0$$

[1] Notice that these demand functions are a special case in which the demand for each commodity depends only upon its own price and income.

[2] If the utility function were quasi-concave but not *strictly* quasi-concave, the indifference curves would possess straight-line portions, and maxima would not need to be unique. In this case more than one value of the quantity demanded may correspond to a given price, and the demand relationship is called a *demand correspondence* rather than a demand function.

Since $k \neq 0$,

$$y^0 - p_1q_1 - p_2q_2 = 0$$

Eliminating k from the first two equations of (2-15) by moving the second terms to the right-hand side and dividing the first equation by the second,

$$\frac{f_1}{f_2} = \frac{p_1}{p_2}$$

The last two equations are the same as (2-7) and (2-10). Therefore the demand function for the price-income set (kp_1, kp_2, ky^0) is derived from the same equations as for the price-income set (p_1, p_2, y^0). It is also easy to demonstrate that the second-order conditions are unaffected. This proves that the demand functions are homogeneous of degree zero in prices and income. If all prices and the consumer's income are increased in the same proportion, the quantities demanded by the consumer do not change. This implies a relevant and empirically testable restriction upon the consumer's behavior; it means that he will not behave as if he were richer (or poorer) in terms of real income if his income and prices rise in the same proportion. A rise in money income is desirable for the consumer, *ceteris paribus*, but its benefits are illusory if prices change proportionately. If such proportionate changes leave his behavior unaltered, there is an absence of "money illusion."

Compensated Demand Functions

Imagine a situation in which some public authority taxes or subsidizes a consumer in such a way as to leave his utility unchanged after a price change. Assume that this is done by providing a lump-sum payment that will give the consumer the minimum income necessary to achieve his initial utility level. The consumer's compensated demand functions give the quantities of the commodities that he will buy as functions of commodity prices under these conditions. They are obtained by minimizing the consumer's expenditures subject to the constraint that his utility is at the fixed level U^0.

Assume again that the utility function is $U = q_1q_2$. Form the expression

$$Z = p_1q_1 + p_2q_2 + \mu(U^0 - q_1q_2)$$

and set its partial derivatives equal to zero:

$$\frac{\partial Z}{\partial q_1} = p_1 - \mu q_2 = 0$$

$$\frac{\partial Z}{\partial q_2} = p_2 - \mu q_1 = 0$$

$$\frac{\partial Z}{\partial \mu} = U^0 - q_1q_2 = 0$$

Solving for q_1 and q_2 gives the compensated demand functions:

$$q_1 = \sqrt{\frac{U^0 p_2}{p_1}} \qquad q_2 = \sqrt{\frac{U^0 p_1}{p_2}}$$

The reader can easily verify that these functions are homogeneous of degree zero in prices.

Demand Curves

In general, the consumer's ordinary demand function for Q_1 is written as

$$q_1 = \phi(p_1, p_2, y^0)$$

or, assuming that p_2 and y^0 are given parameters,

$$q_1 = D(p_1)$$

It is often assumed that the demand function possesses an inverse such that price may be expressed as a unique function of quantity. An *inverse* demand function $p_1 = D^{-1}(q_1)$ is a function such that $D[D^{-1}(q_1)] = q_1$.

The shape of the demand function depends upon the properties of the consumer's utility function. It is generally assumed that demand curves are negatively sloped: the lower the price, the greater the quantity demanded. In exceptional cases the opposite relationship may hold. An example is provided by ostentatious consumption: If the consumer derives utility from a high price, the demand function may have a positive slope. The nature of price-induced changes in the quantity demanded is analyzed in detail in Sec. 2-5. Elsewhere in this volume it is assumed that demand functions are negatively sloped.

The consumer's compensated demand curve for Q_1 is constructed in a similar fashion with p_2 and U^0 as given parameters. In Sec. 2-5 it is shown that the convexity of the indifference curves ensures that compensated demand curves are always downward sloping.

Possible shapes for ordinary and compensated demand curves are shown in Fig. 2-5. The ordinary demand curve is labeled *DD* and the compensated demand curve is labeled *D'D'*. The values at their point of intersection, p_1^0 and q_1^0, satisfy both functions. At this point the utility level achieved for the ordinary demand curve equals the level prescribed for the compensated demand curve, and the minimum income for the compensated demand curve equals the fixed income for the ordinary demand curve. At prices greater than p_1^0 income compensation will be positive, and the compensated demand curve will yield higher quantities for each price. At prices less than p_1^0 income compensation will be negative, and the compensated demand curve will yield lower quantities for each price.

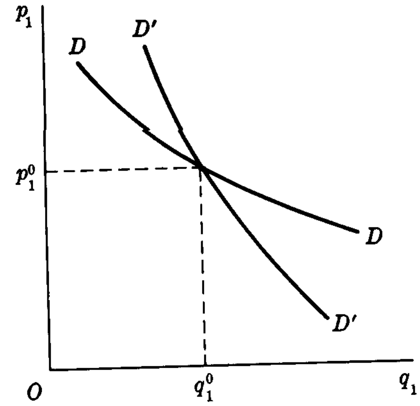

Figure 2-5

Price and Income Elasticities of Demand

The own elasticity of demand for $Q_1(\varepsilon_{11})$ is defined as the proportionate rate of change of q_1 divided by the proportionate rate of change of its own price with p_2 and y^0 constant:

$$\varepsilon_{11} = \frac{\partial(\ln q_1)}{\partial(\ln p_1)} = \frac{p_1}{q_1}\frac{\partial q_1}{\partial p_1} \tag{2-16}$$

A numerically large value for an elasticity implies that quantity is proportionately very responsive to price changes. Commodities which have numerically high elasticities ($\varepsilon_{11} < -1$) are often called luxuries, whereas those with numerically small elasticities ($\varepsilon_{11} > -1$) are called necessities. Price elasticities of demand are pure numbers independent of the units in which prices and outputs are measured. The elasticity ε_{11} is negative if the corresponding demand curve is downward sloping.

The consumer's expenditure on Q_1 is p_1q_1, and

$$\frac{\partial(p_1q_1)}{\partial p_1} = q_1 + p_1\frac{\partial q_1}{\partial p_1} = q_1\left(1 + \frac{p_1}{q_1}\frac{\partial q_1}{\partial p_1}\right) = q_1(1 + \varepsilon_{11})$$

The consumer's expenditures on Q_1 will increase with p_1 if $\varepsilon_{11} > -1$, remain unchanged if $\varepsilon_{11} = -1$, and decrease if $\varepsilon_{11} < -1$.

A cross-price elasticity of demand for the ordinary demand function relates the proportionate change in one quantity to the proportionate change in the other price. For example,

$$\varepsilon_{21} = \frac{\partial(\ln q_2)}{\partial(\ln p_1)} = \frac{p_1}{q_2}\frac{\partial q_2}{\partial p_1} \tag{2-17}$$

Cross-price elasticities may be either positive or negative.

Taking the total differential of the budget constraint (2-7) and letting $dy^0 = dp_2 = 0$,

$$p_1\,dq_1 + q_1\,dp_1 + p_2\,dq_2 = 0$$

Multiplying through by $p_1q_1q_2/y^0q_1q_2\,dp_1$, and rearranging terms,

$$\alpha_1\varepsilon_{11} + \alpha_2\varepsilon_{21} = -\alpha_1 \tag{2-18}$$

where $\alpha_1 = p_1q_1/y^0$ and $\alpha_2 = p_2q_2/y^0$ are the proportions of total expenditures for the two goods. Equation (2-18) is called the Counnot aggregation condition. If the own-price elasticity of demand for Q_1 is known, (2-18) can be used to evaluate the cross-price elasticity of demand for Q_2. If $\varepsilon_{11} = -1$, $\varepsilon_{21} = 0$. If $\varepsilon_{11} < -1$, $\varepsilon_{21} > 0$, and if $\varepsilon_{11} > -1$, $\varepsilon_{21} < 0$.

Own- and cross-price elasticities of demand for compensated demand functions can be defined in an analogous manner by inserting compensated rather than ordinary demand functions in (2-15) and (2-17). Equation (2-18) does not hold for compensated demand functions. Taking the total differential of the utility function (2-1) and letting $dU = 0$,

$$f_1\,dq_1 + f_2\,dq_2 = 0$$

Using the first-order condition $p_1/p_2 = f_1/f_2$, multiplying through by $p_1q_1q_2/y^0q_1q_2\,dp_1$, and rearranging terms,

$$\alpha_1\xi_{11} + \alpha_2\xi_{21} = 0 \tag{2-19}$$

where the compensated price elasticities are denoted by ξ_{11} and ξ_{21}. Since $\xi_{11} < 0$, it follows from (2-19) that $\xi_{21} > 0$.

Returning to the example $U = q_1q_2$, the own- and cross-price elasticities for the ordinary demand function are

$$\varepsilon_{11} = -\frac{p_1}{q_1}\frac{y^0}{2p_1^2} = -\frac{p_1}{y^0/2p_1}\frac{y^0}{2p_1^2} = -1$$

$$\varepsilon_{21} = \frac{p_1}{q_2}0 = 0$$

This is a special case. Not all demand functions have unit own and zero cross elasticities or even constant elasticities. In general, elasticities are a function of p_1, p_2, and y^0. The reader can verify that the compensated elasticities for this example are $\xi_{11} = -\frac{1}{2}$ and $\xi_{21} = \frac{1}{2}$.

An income elasticity of demand for an ordinary demand function is defined as the proportionate change in the purchases of a commodity relative to the proportionate change in income with prices constant:

$$\eta_1 = \frac{\partial(\ln q_1)}{\partial(\ln y)} = \frac{y}{q_1}\frac{\partial\phi(p_1, p_2, y)}{\partial y} \tag{2-20}$$

where η_1 denotes the income elasticity of demand for q_1. Income elasticities can be positive, negative, or zero, but are normally assumed to be positive.

Taking the total differential of the budget constraint (2-7),

$$p_1\,dq_1 + p_2\,dq_2 = dy$$

Multiplying through by y/y, multiplying the first term on the left by q_1/q_1, the second by q_2/q_2, and dividing through by dy,

$$\alpha_1\eta_1 + \alpha_2\eta_2 = 1 \tag{2-21}$$

which is called the Engel aggregation condition. The sum of the income elasticities weighted by total expenditure proportions equals unity. Income elasticities cannot be derived for compensated demand functions since income is not an argument of these functions.

2-4 INCOME AND LEISURE

If the consumer's income is payment for work performed by him, the optimum amount of work that he performs can be derived from the analysis of utility maximization. One can also derive the consumer's demand curve for income from this analysis. Assume that the consumer's satisfaction depends on income and leisure. His utility function is

$$U = g(L, y) \tag{2-22}$$

where L denotes leisure. Both income and leisure are desirable. In the preceding sections it is assumed that the consumer derives utility from the commodities he purchases with his income. In the construction of (2-22) it is assumed that he buys the various commodities at constant prices, and income is thereby treated as generalized purchasing power (see Sec. 3-6).

The rate of substitution of income for leisure is

$$-\frac{dy}{dL} = \frac{g_1}{g_2}$$

Denote the amount of work performed by the consumer by W and the wage rate by r. By definition,

$$L = T - W \tag{2-23}$$

where T is the total amount of available time.[1] The budget constraint is

$$y = rW \tag{2-24}$$

Substituting (2-23) and (2-24) into (2-22),

$$U = g(T - W, rW) \tag{2-25}$$

To maximize utility set the derivative of (2-25) with respect to W equal to zero:[2]

$$\frac{dU}{dW} = -g_1 + g_2 r = 0$$

[1] For example, if the period for which the utility function is defined is one day, $T = 24$ hours.
[2] The composite-function rule is employed.

and therefore
$$-\frac{dy}{dL} = \frac{g_1}{g_2} = r \tag{2-26}$$

which states that the rate of substitution of income for leisure equals the wage rate. The second-order condition states

$$\frac{d^2U}{dW^2} = g_{11} - 2g_{12}r + g_{22}r^2 < 0$$

Equation (2-26) is a relation in terms of W and r and is based on the individual consumer's optimizing behavior. It is therefore the consumer's supply curve for work and states how much he will work at various wage rates. Since the supply of work is equivalent to the demand for income, (2-26) indirectly provides the consumer's demand curve for income.

Assume that the utility function, defined for a time period of one day, is given by $U = 48L + Ly - L^2$. Then

$$U = 48(T - W) + (T - W)Wr - (T - W)^2$$

and setting the derivative equal to zero,

$$\frac{dU}{dW} = -48 - Wr + r(T - W) + 2(T - W) = 0$$

Therefore
$$W = \frac{T(r+2) - 48}{2(r+1)}$$

and y may be obtained by substituting in (2-24). The second-order condition is fulfilled, since

$$\frac{d^2U}{dW^2} = -2(r+1) < 0$$

for any positive wage. In the present case the individual's supply function has the following characteristics:

1. Since T, the total available time, is 24 (hours), at a zero wage the individual will not work at all.
2. Since dW/dr is positive, hours worked will increase with the wage.
3. Irrespective of how high the wage becomes, the individual will never work more than 12 hours per day, since $\lim_{r \to \infty} W = 12$.

2-5 SUBSTITUTION AND INCOME EFFECTS

The Slutsky Equation

Comparative statics analysis examines the effect of perturbations in exogenous variables (such as prices and incomes in the present case) on the solution values for the endogenous variables (namely, quantities). Changes in prices and income will normally alter the consumer's expenditure pattern, but the new quantities (and prices and income) will always satisfy the first-order conditions (2-9). In order to find the magnitude of the effect of price and income changes on the consumer's purchases, allow all variables to vary simultaneously. This is accomplished by total differentiation of Eqs. (2-9):

$$\begin{aligned} f_{11}\,dq_1 + f_{12}\,dq_2 - p_1\,d\lambda &= \lambda\,dp_1 \\ f_{21}\,dq_1 + f_{22}\,dq_2 - p_2\,d\lambda &= \lambda\,dp_2 \\ -p_1\,dq_1 - p_2\,dq_2 \qquad &= -dy + q_1\,dp_1 + q_2\,dp_2 \end{aligned} \tag{2-27}$$

In order to solve this system of three equations for the three unknowns, dq_1, dq_2, and $d\lambda$, the terms on the right must be regarded as constants. The array of coefficients formed by (2-27) is the same as the bordered Hessian determinant (2-12). Denoting this determinant by $\mathcal{D}$ and the cofactor of the element in the first row and the first column by $\mathcal{D}_{11}$, the cofactor of the element in the first row and second column by $\mathcal{D}_{12}$, etc., the solution of (2-27) by Cramer's rule (see Sec. A-1) is

$$dq_1 = \frac{\lambda\mathcal{D}_{11}\,dp_1 + \lambda\mathcal{D}_{21}\,dp_2 + \mathcal{D}_{31}(-dy + q_1\,dp_1 + q_2\,dp_2)}{\mathcal{D}} \tag{2-28}$$

$$dq_2 = \frac{\lambda\mathcal{D}_{12}\,dp_1 + \lambda\mathcal{D}_{22}\,dp_2 + \mathcal{D}_{32}(-dy + q_1\,dp_1 + q_2\,dp_2)}{\mathcal{D}} \tag{2-29}$$

Dividing both sides of (2-28) by dp_1 and assuming that p_2 and y do not change ($dp_2 = dy = 0$),

$$\frac{\partial q_1}{\partial p_1} = \frac{\mathcal{D}_{11}\lambda}{\mathcal{D}} + q_1\frac{\mathcal{D}_{31}}{\mathcal{D}} \tag{2-30}$$

The partial derivative on the left-hand side of (2-30) is the rate of change of the consumer's purchases of Q_1 with respect to changes in p_1, all other things being equal. *Ceteris paribus*, the rate of change with respect to income is

$$\frac{\partial q_1}{\partial y} = -\frac{\mathcal{D}_{31}}{\mathcal{D}} \tag{2-31}$$

Changes in commodity prices change the consumer's level of satisfaction, since a new equilibrium is established which lies on a different indifference curve.

Consider a price change that is compensated by an income change that leaves the consumer on his initial indifference curve. An increase in the price of a commodity is accompanied by a corresponding increase in his income such that $dU = 0$ and $f_1\,dq_1 + f_2\,dq_2 = 0$ by (2-3). Since $f_1/f_2 = p_1/p_2$, it is also true that $p_1\,dq_1 + p_2\,dq_2 = 0$. Hence, from the last equation of (2-27), $-dy + q_1\,dp_1 + q_2\,dp_2 = 0$, and

$$\left(\frac{\partial q_1}{\partial p_1}\right)_{U=\text{const}} = \frac{\mathcal{D}_{11}\lambda}{\mathcal{D}} \tag{2-32}$$

Equation (2-30) can now be rewritten as

$$\frac{\partial q_1}{\partial p_1} = \left(\frac{\partial q_1}{\partial p_1}\right)_{U=\text{const}} - q_1 \left(\frac{\partial q_1}{\partial y}\right)_{\text{prices}=\text{const}} \tag{2-33}$$

Equation (2-33) is known as the *Slutsky equation.* The quantity $\partial q_1/\partial p_1$ is the slope of the ordinary demand curve for Q_1, and the first term on the right is the slope of the compensated demand curve for Q_1.

An alternative compensation criterion is that the consumer is provided enough income to purchase his former consumption bundle so that $dy = q_1\, dp_1 + q_2\, dp_2$. This is the equation that led to (2-32). Here

$$\left(\frac{\partial q_1}{\partial p_1}\right)_{q_1,\, q_2=\text{const}} = \frac{\mathscr{D}_{11}\lambda}{\mathscr{D}}$$

which can be substituted for the first term on the right of (2-33). At first glance it might appear remarkable that two rather different compensation schemes led to the same result. However, they only define the same derivative, and may lead to quite different results for any finite move. A consumer can be induced to stay on the same indifference curve in the finite case, but he cannot be induced to purchase the same bundle if relative prices change. All subsequent analysis here is based upon (2-33).

The Slutsky equation may be expressed in terms of the price and income elasticities described in Sec. 2-3. Multiplying (2-33) through by p_1/q_1 and multiplying the last term on the right by y/y,

$$\varepsilon_{11} = \xi_{11} - \alpha_1 \eta_1 \tag{2-34}$$

The price elasticity of the ordinary demand curve equals the price elasticity of the compensated demand curve less the corresponding income elasticity multiplied by the proportion of total expenditures spent on Q_1. Hence, the ordinary demand curve will have a greater demand elasticity than the compensated demand curve; that is, ε_{11} will be more negative than ξ_{11} if the income elasticity of demand is positive.

Direct Effects

The first term on the right-hand side of (2-33) is the *substitution effect,* or the rate at which the consumer substitutes Q_1 for other commodities when the price of Q_1 changes and he moves along a given indifference curve.[1] The second term on the right is the *income effect,* which states the rate at which the consumer's purchases of Q_1 would change with changes in his income, prices remaining constant. The sum of the two rates gives the total rate of change for Q_1 as p_1 changes.

In the present case the multiplier λ is the derivative of utility with respect to income with prices constant and quantities variable. From the utility

[1] Slutsky called this the *residual variability* of the commodity in question.

function (2-1) it follows that $\partial U/\partial y = f_1(\partial q_1/\partial y) + f_2(\partial q_2/\partial y)$. Substituting $f_1 = \lambda p_1$ and $f_2 = \lambda p_2$,

$$\frac{\partial U}{\partial y} = \lambda\left(p_1\frac{\partial q_1}{\partial y} + p_2\frac{\partial q_2}{\partial y}\right) = \lambda$$

which follows from the partial derivative of the budget constraint (2-7) with respect to $y: 1 = p_1(\partial q_1/\partial y) + p_2(\partial q_2/\partial y)$. This confirms the result inferred from (2-11) at an earlier stage.

Solving (2-27) for $d\lambda$,

$$d\lambda = \frac{\lambda \mathscr{D}_{13}\,dp_1 + \lambda \mathscr{D}_{23}\,dp_2 + \mathscr{D}_{33}(-dy + q_1\,dp_1 + q_2\,dp_2)}{\mathscr{D}} \tag{2-35}$$

Assume now that only income changes, i.e., that $dp_1 = dp_2 = 0$. Then (2-35) becomes

$$\frac{\partial \lambda}{\partial y} = -\frac{\mathscr{D}_{33}}{\mathscr{D}} = -\frac{f_{11}f_{22} - f_{12}^2}{\mathscr{D}} \tag{2-36}$$

Since $\mathscr{D}$ is positive, the rate of change of the marginal utility of income will have the same sign as $-(f_{11}f_{22} - f_{12}^2)$. This would be negative if the utility function were strictly concave. However, for ordinal utility functions only strict quasi-concavity is assumed, and the theory does not predict whether the marginal utility of income is increasing or decreasing with income.

By (2-32) the substitution effect is $\mathscr{D}_{11}\lambda/\mathscr{D}$. The determinant $\mathscr{D}$, which is the same as (2-12), is positive. Expanding $\mathscr{D}_{11}$,

$$\mathscr{D}_{11} = -p_2^2$$

which is clearly negative. This proves that the sign of the substitution effect is always negative and that the compensated demand curve is always downward sloping.

A change in real income may cause a reallocation of the consumer's resources even if prices do not change or if they change in the same proportion. The income effect is $-q_1(\partial q_1/\partial y)_{\text{prices=const}}$ and may be of either sign. The final effect of a price change on the purchases of the commodity is thus unknown. However, an important conclusion can still be derived: The smaller the quantity of Q_1, the less significant is the income effect. A commodity Q_1 is called an *inferior good* if the consumer's purchases decrease as income rises and increase as income falls; i.e., if $\partial q_1/\partial y$ is negative, which makes the income effect positive. A *Giffen good* is an inferior good with an income effect large enough to offset the negative substitution effect and make $\partial q_1/\partial p_1$ positive. This means that as the price of Q_1 falls, the consumer's purchases of Q_1 will also fall. This may occur if a consumer is sufficiently poor so that a considerable portion of his income is spent on a commodity such as potatoes which he needs for his subsistence. Assume now that the price of potatoes falls. The consumer who is not very fond of potatoes may suddenly discover that his real income has increased as a result of the price fall. He will then buy fewer potatoes and purchase a more palatable diet with the remainder of his income.

The Slutsky equation can be derived for the specific utility function assumed in the previous examples. State the budget constraint in the general implicit form $y - p_1q_1 - p_2q_2 = 0$, and form the function

$$V = q_1q_2 + \lambda(y - p_1q_1 - p_2q_2)$$

Setting the partial derivatives equal to zero,

$$q_2 - \lambda p_1 = 0$$
$$q_1 - \lambda p_2 = 0$$
$$y - p_1q_1 - p_2q_2 = 0$$

The total differentials of these equations are

$$\begin{aligned} dq_2 - p_1\,d\lambda &= \lambda\,dp_1 \\ dq_1 - p_2\,d\lambda &= \lambda\,dp_2 \\ -p_1\,dq_1 - p_2\,dq_2 &= -dy + q_1\,dp_1 + q_2\,dp_2 \end{aligned}$$

Denote the determinant of the coefficients of these equations by $\mathcal{D}$ and the cofactor of the element in the ith row and jth column by $\mathcal{D}_{ij}$. Simple calculations show that

$$\mathcal{D} = 2p_1p_2$$
$$\mathcal{D}_{11} = -p_2^2$$
$$\mathcal{D}_{21} = p_1p_2$$
$$\mathcal{D}_{31} = -p_2$$

Solving for dq_1 by Cramer's rule gives

$$dq_1 = \frac{-p_2^2\lambda\,dp_1 + p_1p_2\lambda\,dp_2 - p_2(-dy + q_1\,dp_1 + q_2\,dp_2)}{2p_1p_2}$$

Assuming that only the price of the first commodity varies,

$$\frac{\partial q_1}{\partial p_1} = -\frac{p_2\lambda}{2p_1} - \frac{q_1}{2p_1}$$

The value of λ is obtained by substituting the values of q_1 and q_2 from the first two equations of the first-order conditions into the third and solving for λ in terms of the parameters p_1, p_2, and y. Thus $\lambda = y/2p_1p_2$. Substituting this value into the above equation and then introducing into it the values of the parameters ($y = 100$, $p_1 = 2$, $p_2 = 5$) and also the equilibrium value of q_1 (25), a numerical answer is obtained:

$$\frac{\partial q_1}{\partial p_1} = -12.5$$

The meaning of this answer is the following: If, starting from the initial equilibrium situation, p_1 were to change, *ceteris paribus*, the consumer's purchases would change at the rate of 12.5 units of Q_1 per dollar of change in the price of Q_1; furthermore the direction of the change in the consumer's purchases is opposite to the direction of the price change. The expression $-p_2\lambda/2p_1$ is the substitution effect, and its value in the present example is -6.25. The expression $-q_1/2p_1$ is the income effect, also with a value of -6.25.

Cross Effects

The Slutsky equation (2-33) and its elasticity representation (2-34) can be extended to account for changes in the demand for one commodity resulting from changes in the price of the other. The generalized forms are

$$\frac{\partial q_i}{\partial p_j} = \frac{\mathcal{D}_{ji}\lambda}{\mathcal{D}} + q_j \frac{\mathcal{D}_{3i}}{\mathcal{D}} = \left(\frac{\partial q_i}{\partial p_j}\right)_{U=\text{const}} - q_j \left(\frac{\partial q_i}{\partial y}\right)_{\text{prices}=\text{const}} \tag{2-37}$$

and

$$\varepsilon_{ij} = \xi_{ij} - \alpha_j \eta_i \tag{2-38}$$

for $i, j = 1, 2$. The signs of the cross-substitution effects $(i \neq j)$ are not known in general. Let $S_{ij} = \mathcal{D}_{ji}\lambda/\mathcal{D}$ denote the substitution effect when the quantity of the ith commodity is adjusted as a result of a variation in the jth price. Since $\mathcal{D}$ is a symmetric determinant,[1] $\mathcal{D}_{12} = \mathcal{D}_{21}$, and it follows that $S_{ij} = S_{ji}$. The substitution effect on the ith commodity resulting from a change in the jth price is the same as the substitution effect on the jth commodity resulting from a change in the ith price.

This is a remarkable conclusion. Imagine that the consumer's demand for tea increases at the rate of 2 cups of tea per 1-cent increase in the price of coffee. One can infer from this that his purchases of coffee would increase at the rate of 2 cups of coffee per 1-cent increase in the price of tea.

Sum the compensated demand elasticities for Q_1 as a result of changes in p_1 and p_2:

$$\xi_{11} + \xi_{12} = \frac{p_1\mathcal{D}_{11}\lambda}{q_1\mathcal{D}} + \frac{p_2\mathcal{D}_{21}\lambda}{q_1\mathcal{D}} = \frac{\lambda(p_1\mathcal{D}_{11} + p_2\mathcal{D}_{21})}{q_1\mathcal{D}} = 0 \tag{2-39}$$

The term in parentheses equals zero since it is an expansion of the determinant of (2-27) in terms of alien cofactors; i.e., the cofactors of the elements of the first column are multiplied by the negative of the elements in the last column. Thus, the negative compensated elasticity for Q_1 with respect to p_1 equals in absolute value the positive compensated elasticity for Q_1 with respect to p_2.

Sum the negative of the ordinary demand elasticities for Q_1 as a result of

[1] A determinant is symmetric if its array is symmetric around the principal diagonal.

changes in p_1 and p_2 as given by (2-38):

$$-(\varepsilon_{11}+\varepsilon_{12}) = -(\xi_{11}+\xi_{12}) + (\alpha_1+\alpha_2)\eta_1 = \eta_1 \tag{2-40}$$

from (2-39) and $\alpha_1+\alpha_2 = 1$. The income elasticity of demand for a commodity equals the negative of the sum of ordinary price elasticities of demand for that commodity with respect to its own and the other price.

Substitutes and Complements

Two commodities are substitutes if both can satisfy the same need of the consumer; they are complements if they are consumed jointly in order to satisfy some particular need. These are loose definitions, but everyday experience may suggest some plausible examples. Coffee and tea are most likely substitutes, whereas coffee and sugar are most likely complements. A more rigorous definition of substitutability and complementarity is provided by the cross-substitution term of the Slutsky equation (2-37). Accordingly, Q_1 and Q_2 are substitutes if the substitution effect $\mathscr{D}_{21}\lambda/\mathscr{D}$ is positive; they are complements if it is negative. If Q_1 and Q_2 are substitutes (in the everyday sense) and if compensating variations in income keep the consumer on the same indifference curve, an increase in the price of Q_1 will induce the consumer to substitute Q_2 for Q_1. Then $(\partial q_2/\partial p_1)_{U=\text{const}} > 0$. For analogous reasons, $(\partial q_2/\partial p_1)_{U=\text{const}} < 0$ in the case of complements.[1]

All commodities cannot be complements for each other. Hence only substitutability can occur in the present two-variable case. This theorem is easily proved. Multiply (2-30) by p_1, (2-31) by y, and (2-37) for $i = 1$ and $j = 2$ by p_2, and add:

$$\frac{\mathscr{D}_{11}\lambda}{\mathscr{D}}p_1 + q_1\frac{\mathscr{D}_{31}}{\mathscr{D}}p_1 + \frac{\mathscr{D}_{21}\lambda}{\mathscr{D}}p_2 + q_2\frac{\mathscr{D}_{31}}{\mathscr{D}}p_2 - \frac{\mathscr{D}_{31}}{\mathscr{D}}y$$
$$= \frac{1}{\mathscr{D}}[\mathscr{D}_{11}\lambda p_1 + \mathscr{D}_{21}\lambda p_2 - \mathscr{D}_{31}(y - p_1q_1 - p_2q_2)]$$
$$= \frac{1}{\mathscr{D}}[\mathscr{D}_{11}\lambda p_1 + \mathscr{D}_{21}\lambda p_2 - \mathscr{D}_{31}(0)] = 0$$

The final bracketed term equals zero since it is an expansion in terms of alien cofactors as in (2-39). Substituting $S_{ij} = \mathscr{D}_{ji}\lambda/\mathscr{D}$,

$$S_{11}p_1 + S_{12}p_2 = 0 \tag{2-41}$$

The substitution effect for Q_1 resulting from changes in p_1, S_{11}, is known to be negative. Hence (2-41) implies that S_{12} must be positive, and in terms of the definitions of substitutability and complementarity this means that Q_1 and Q_2 are necessarily substitutes.

[1] This provides a rationale for the definitions. When $(\partial q_2/\partial p_1)_{U=\text{const}} = 0$, Q_1 and Q_2 are independent.

Commodities i and j are ***gross substitutes*** or ***gross complements*** according to whether the total effect $\partial q_i/\partial p_j$ is positive or negative. In the two-good case it is possible for a pair of goods to be substitutes in terms of S_{ij}, and at the same time to be gross complements. In the n-good case it is also possible for them to be complements in terms of the S_{ij}, and at the same time be gross substitutes. A two-good example is provided by the utility function $U = q_1q_2 - q_2$ with the domain $q_1 > 1, q_2 > 0$. Maximizing subject to the budget constraint yields the demand function $q_2 = (y - p_1)/2p_2$, which is valid for the domain $p_1 < y$. Here $\partial q_2/\partial p_1 < 0$ to make the commodities gross complements even though they are substitutes in terms of S_{12}.

2-6 GENERALIZATION TO n VARIABLES

The foregoing analysis of the consumer is now generalized to the case of n commodities. The generalization is not carried out in detail, but the first few steps are indicated. If there are n commodities, the utility function is

$$U = f(q_1, q_2, \ldots, q_n)$$

and the budget constraint is given by

$$y - \sum_{i=1}^{n} p_iq_i = 0$$

Forming the Lagrange function as above,

$$V = f(q_1, q_2, \ldots, q_n) + \lambda \left(y - \sum_{i=1}^{n} p_iq_i\right)$$

Setting the partial derivatives equal to zero,

$$\frac{\partial V}{\partial q_i} = f_i - \lambda p_i = 0 \qquad i = 1, \ldots, n \tag{2-42}$$

Conditions (2-42) can be modified to state the equality for all commodities of marginal utility divided by price. The partial derivative of V with respect to λ is again the budget constraint. There are a total of $(n + 1)$ equations in $(n + 1)$ variables (n q's and λ). The demand curves for the n commodities can be obtained by solving for the q's. Conditions (2-42) can be stated alternatively as

$$-\frac{\partial q_i}{\partial q_j} = \frac{p_j}{p_i}$$

for all i and j; i.e., the rate of commodity substitution of commodity i for commodity j must equal the price ratio p_j/p_i. Second-order conditions must be fulfilled in order to ensure that a batch of commodities that satisfies (2-42) is optimal. The bordered Hessian determinants must alternate in sign:

$$\begin{vmatrix} f_{11} & f_{12} & -p_1 \\ f_{21} & f_{22} & -p_2 \\ -p_1 & -p_2 & 0 \end{vmatrix} > 0, \qquad \begin{vmatrix} f_{11} & f_{12} & f_{13} & -p_1 \\ f_{21} & f_{22} & f_{23} & -p_2 \\ f_{31} & f_{32} & f_{33} & -p_3 \\ -p_1 & -p_2 & -p_3 & 0 \end{vmatrix} < 0,$$

$$\ldots, (-1)^n \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} & -p_1 \\ f_{21} & f_{22} & \cdots & f_{2n} & -p_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} & -p_n \\ -p_1 & -p_2 & \cdots & -p_n & 0 \end{vmatrix} > 0$$

which is a generalization of condition (2-12).

The assumption of convexity for indifference curves in two dimensions may also be extended to indifference hypersurfaces in n dimensions. The first of the second-order conditions for the n-dimensional case is the same as the second-order condition for the two-dimensional case, which was demonstrated to result in a decreasing RCS between the commodities. In n dimensions the second-order conditions result in decreasing RCSs between every pair of commodities. The satisfaction of the second-order conditions is ensured by the regular strict quasi-concavity of the utility function.

Other theorems can also be generalized in straightforward fashion. The Slutsky equations (2-37) and (2-38) hold for $i, j = 1, \ldots, n$. The generalization of (2-41) is

$$\sum_{j=1}^{n} S_{ij} p_j = 0 \qquad i = 1, \ldots, n \tag{2-43}$$

It still follows that all commodities cannot be complements for each other. However, some pairs of commodities can be complements; i.e., some $S_{ij} < 0$ for $i \neq j$. Gross substitutes and complements may also be defined.

All the elasticity relations generalize. The n-commodity forms for (2-18), (2-19), and (2-21) respectively are

$$\sum_{i=1}^{n} \alpha_i \varepsilon_{ij} = -\alpha_j \qquad j = 1, \ldots, n$$

$$\sum_{i=1}^{n} \alpha_i \xi_{ij} = 0 \qquad j = 1, \ldots, n$$

$$\sum_{j=1}^{n} \alpha_j \eta_j = 1$$

and the general forms for (2-39) and (2-40) respectively are

$$\sum_{j=1}^{n} \xi_{ij} = 0 \qquad i = 1, \ldots, n$$

$$-\sum_{j=1}^{n} \varepsilon_{ij} = \eta_i \qquad i = 1, \ldots, n$$

2-7 SUMMARY

Nineteenth-century economic theorists explained the consumer's behavior on the assumption that utility is measurable. This restrictive assumption was abandoned around the turn of the last century, and the consumer was assumed to be capable only of ranking commodity combinations consistently in order of preference. This ranking is described mathematically by the consumer's ordinal utility function, which always assigns a higher number to a more desirable combination of commodities. The consumer is normally assumed to have a regular strictly quasi-concave utility function which implies a decreasing rate of commodity substitution (RCS).

The basic postulate of the theory of consumer behavior is that the consumer maximizes utility. Since his income is limited, he maximizes utility subject to a budget constraint, which expresses his income limitation in mathematical form. The consumer's RCS must equal the price ratio for a maximum. In diagrammatic terms, the optimum commodity combination is given by the point at which his income line is tangent to an indifference curve. The second-order condition for a maximum is guaranteed by the convexity assumption.

The consumer's utility function is not unique. If a particular function describes appropriately the consumer's preferences, so does any other function which is a positive monotonic transformation of the first. Other kinds of transformations do not preserve the correct ranking, and the utility function is unique up to a positive monotonic transformation.

The consumer's ordinary demand functions for commodities can be derived from his first-order conditions for utility maximization. These state quantities demanded as functions of all prices and the consumer's income. Ordinary demand functions are single-valued and homogeneous of degree zero in prices and income: a proportionate change in all prices and the consumer's income leaves the quantity demanded unchanged. The consumer's compensated demand functions for commodities are constructed on the assumption that his income is increased or decreased following a price change in order to leave him at his initial utility level. The compensated demand functions state quantities demanded as functions of all prices. They are single-valued and homogeneous of degree zero in prices. A demand curve is obtained by stating quantity demanded as a function of own price on the assumption that the other arguments of the demand function are given parameters. Price elasticities are defined for both types of demand functions, and income elasticities are defined for ordinary demand functions.

In general, the amount of labor performed by a consumer affects his level of utility. The amount of labor performed by the consumer can be determined on the basis of the rational-decision criterion of utility maximization. The equilibrium conditions are similar to those which hold for the selection of an optimal commodity combination.

The consumer's reaction to price and income changes can be analyzed in

terms of substitution and income effects. The effect of a given price change can be analytically decomposed into a substitution effect, which measures the rate at which he would substitute commodities for each other by moving along the same indifference curve, and an income effect as a residual category. If the price of a commodity changes, the quantity demanded changes in the opposite direction if the consumer is forced to move along the same indifference curve: the substitution effect is negative. If the income effect is positive, the commodity is an inferior good. If the total effect is positive, it is also a Giffen good. Substitutes and complements are defined in terms of the sign of the substitution effect for one commodity when the price of another changes: a positive cross-substitution effect means substitutability, and a negative one, complementarity. Gross substitutes and complements are defined in terms of the full effect of price changes on quantity. In conclusion, the generalization of the theory to n commodities is indicated.

EXERCISES

2-1 Determine whether the following utility functions are regular strictly quasi-concave for the domain $q_1 > 0$, $q_2 \geq 0$: $U = q_1 q_2$; $U = q_1^\gamma q_2$; $U = q_1^2 + q_2^2$; $U = q_1 + q_2 + 2q_1 q_2$; $U = q_1 q_2 - 0.01(q_1^2 + q_2^2)$; and $U = q_1 q_2 + q_1 q_3 + q_2 q_3$.

2-2 Let $f(q_1, q_2)$ be a strictly concave utility function, and let $q_j^{(2)} = (q_j^0 + q_j^{(1)})/2$, $j = 1, 2$, where superscripts denote particular values for the variables. Prove that

$$f(q_1^{(2)}, q_2^{(2)}) - f(q_1^0, q_2^0) > f(q_1^{(1)}, q_2^{(1)}) - f(q_1^{(2)}, q_2^{(2)})$$

2-3 Find the optimum commodity purchases for a consumer whose utility function and budget constraint are $U = q_1^{1.5} q_2$ and $3q_1 + 4q_2 = 100$ respectively.

2-4 The locus of points of tangency between income lines and indifference curves for given prices p_1, p_2 and a changing value of income is called an income expansion line or Engel curve. Show that the Engel curve is a straight line if the utility function is given by $U = q_1^\gamma q_2$, $\gamma > 0$.

2-5 Show that the utility functions $U = Aq_1^\alpha q_2^\beta$ and $W = q_1^{\alpha/\beta} q_2$ are monotonic transformations of each other where A, α, and β are positive.

2-6 Let a consumer's utility function be $U = q_1^6 q_2^4 + 1.5 \ln q_1 + \ln q_2$ and his budget constraint $3q_1 + 4q_2 = 100$. Show that his optimum commodity bundle is the same as in Exercise 2-3. Why is this the case?

2-7 Construct ordinary and compensated demand functions for Q_1 for the utility function $U = 2q_1 q_2 + q_2$. Construct expressions for ε_{11}, ξ_{12}, and η_1.

2-8 Derive the elasticity of supply of work with respect to the wage rate for the supply curve for work given by the example in Sec. 2-4.

2-9 Prove that Q_1 and Q_2 cannot both be inferior goods.

2-10 Verify that $S_{11}p_1 + S_{12}p_2 = 0$ for the utility function $U = q_1^\gamma q_2$.

2-11 Let $U = f(q, H)$ be a utility function the arguments of which are the quantity of a commodity (q) and the time taken to consume it (H). The marginal utilities of both arguments are positive. Let W be the amount of work performed, $W + H = 24$ (hours), r be the wage, and p be the price of q. Formulate the appropriate constrained utility maximization problem. Find an expression for dH/dr. Is its sign determined unambiguously?

2-12 Imagine that coupon rationing is in effect so that each commodity has two prices: a dollar price and a ration-coupon price. Assume that there are three commodities and that the consumer has a dollar income y and a ration-coupon allotment z. Also assume that this allotment is not so liberal that any commodity combination that he can afford to purchase with his dollar income can also be purchased with his coupons. Formulate his constrained-utility-maximization problem assuming a strictly concave utility function. Derive conditions for a maximum. Interpret the conditions from an economic point of view. Find a sufficient condition which guarantees that the imposition of rationing does not alter the consumer's purchases.

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