§R.1 : Polynomials

Basic Definitions:

A polynomial is a term or finite sum of terms in which all variables have whole number exponents

e.g. $5x^4 + 2x^3 + 6x$

Each piece of the expression are its terms (i.e. $5x^4$)

Here, $\underbrace{5}_{\text{coeff.}}x^{\overset{\text{exponent}}{4}}$ has a coefficient of 5 and exponent of 4

Terms having the same variable and exponent are like terms, if not they are called unlike terms

e.g. $\underbrace{5x^4 + 2x^4}_{\text{like terms}} + 3x^3 = 7x^4 + 3x^3$ (unlike terms: $2x^4$, $3x^3$)

$6x^4 + 6p^4$ are unlike terms as well (different variables)

Order of Operations (PEMDAS)

When simplifying an algebraic expression follow this order

Parentheses/Powers (exponents): Start evaluating parenthesis and then powers (exponents) from left to right

Multiplication/Division: again perform in order from left to right

Addition/Subtraction: in order from left to right

e.g. $(6(2+1)^2 + 3(2) - 22)^2 = (6(3)^2 + 6 - 22)^2 = (6\cdot 9 + 6 - 22)^2$
$= (54 + 6 - 22)^2 = (38)^2 = 1444$

Properties of Real Numbers

$\forall a,b,c \in \mathbb{R}$ (for all Real #'s a, b, c)

1. **Commutative:** $a+b=b+a$ e.g. $2+x = x+2$
   $ab = ba$ → $x\cdot 3 = 3x$

2. **Associative:** $(a+b)+c = a+(b+c)$ e.g. $(2+3)+4 = 2+(3+4) = 9$
   $(ab)c = a(bc)$ → $7x(x) = 7(x\cdot x) = 7x^2$

3. **Distributive:** $a(b+c) = ab+ac$ e.g. $3(x+4) = 3x+12$
   → $-(x-2) = -x+2$

Use these properties to simplify polynomials. Always combine like terms in your final simplified answer

e.g. $(8x^3 + 2x^2 + 3) - (3x^2 + 2) = 8x^3 + 2x^2 + 3 - 3x^2 - 2$
$= 8x^3 - x^2 + 1$

Use the distributive property to multiply Polynomials

Note: $a^M \cdot a^N = a^{M+N}$ (i.e. $x^3 \cdot x^5 = x^{3+5} = x^8$)

e.g. $(3y)(2y-4) = 6y^2 - 12y$

For two binomials (2 term polynomials) use the FOIL method

e.g. $(2x-5)(x+4) = \overset{F}{2x(x)} + \overset{O}{2x(4)} - \overset{I}{5(x)} - \overset{L}{5(4)}$
$= 2x^2 + 3x - 20$

e.g. $(2x-5)(x+4)(x+2) = (2x^2+3x-20)(x+2)$
$= 2x^2\cdot x + 2x^2\cdot 2 + 3x\cdot x + 3x\cdot 2 - 20\cdot x - 20\cdot 2$
$= 2x^3 + 4x^2 + 3x^2 + 6x - 20x - 40$
$= 2x^3 + 7x^2 - 14x - 40$

Remember! $(x+y)^n \neq x^n + y^n$ for $n \neq 1$

§R.2 : Factoring

Using the distributive property to rewrite a polynomial as the product of other, smaller polynomials is called <u>factoring</u>

e.g. $18 = 9\cdot 2 = 3\cdot 3\cdot 2$; $2x^2+2x = 2x(x+2)$

The largest ~~term~~ # that can be factored out is called the <u>greatest common factor</u> (gcf)

e.g. $15m+45 = 15(m+3)$, here 15 is the gcf

$2x^2+x = x(2x+1)$

When factoring a trinomial use FOIL backwards

e.g. $y^2+8y+15$

Since coefficient of y^2 (leading term) is 1 we need 2 numbers whose sum is 15 and product is 8

Product	Sum
$15\cdot 1 = 15$	$15+1 = 16$ ✗
$5\cdot 3 = 15$	$5+3 = 8$ ✓

$\Rightarrow y^2+8y+15 = (y+5)(y+3)$

When the coefficient of the leading term (highest degree) is Not 1 you must also consider all possible factors of the 1st term and proceed by trial and error (we'll see a quicker way later)

<u>Special Factorizations</u>

1. $x^2-y^2 = (x+y)(x-y)$ Difference of Squares
2. $x^2+2xy+y^2 = (x+y)^2$ Perfect Square
3. $x^3-y^3 = (x-y)(x^2+xy+y^2)$ Difference of Cubes
4. $x^3+y^3 = (x+y)(x^2-xy+y^2)$ Sum of Cubes

e.g.) Factoring Polynomials

(A) $64p^2 - 49q^2 = (8p)^2 - (7q)^2 = (8p+7q)(8p-7q)$

(B) x^2+36 is a prime polynomial (cannot factor)

(C) $x^2+12x+36=(x+6)^2$

(D) $x^3-8=x^3-2^3=(x-2)(x^2+2x+4)$

(E) $p^4-1=(p^2+1)(p^2-1)=(p^2+1)(p+1)(p-1)$

When factoring, want each polynomial factor to be prime

§R.3 Rational Expressions

Rational Expressions are quotients of polynomials with nonzero denominators

Properties: $\forall$ mathematical expressions P, Q, R, S w/ $Q \neq 0$, $S \neq 0$

(1) $\frac{P}{Q} = \frac{PS}{QS}$ Fundamental Property

(2) $\frac{P}{Q} + \frac{R}{Q} = \frac{P+R}{Q}$ Addition

(3) $\frac{P}{Q} - \frac{R}{Q} = \frac{P-R}{Q}$ Subtraction

(4) $\frac{P}{Q} \cdot \frac{R}{S} = \frac{PR}{QS}$ Multiplication

(5) $\frac{P}{Q} \div \frac{R}{S} = \frac{P}{Q} \cdot \frac{S}{R}$ $(R \neq 0)$ Division

★ (6) $\frac{a^m}{a^n} = a^{m-n}$ (i.e. $\frac{x^4}{3x} = \frac{1}{3}\frac{x^4}{x} = \frac{1}{3}x^{4-1} = \frac{1}{3}x^3$)

e.g. $\frac{m^2+5m+6}{m+3} \cdot \frac{m}{m^2+3m+2} = \frac{\cancel{(m+2)}\cancel{(m+3)}}{\cancel{m+3}} \cdot \frac{m}{\cancel{(m+2)}(m+1)}$

$= \boxed{\frac{m}{m+1}}$

e.g. $\frac{9p-36}{12} \div \frac{5(p-4)}{18}$ (Invert and Multiply!)

$\Rightarrow \frac{9p-36}{12} \cdot \frac{18}{5(p-4)} = \frac{9\cancel{(p-4)}}{12} \cdot \frac{18}{5\cancel{(p-4)}} = \frac{9 \cdot 18}{12 \cdot 5}$

$= \frac{9 \cdot \cancel{(6)}(3)}{\cancel{6} \cdot 2 \cdot 5} = \frac{27}{5}$

e.g. $\frac{7}{p}+\frac{9}{2p}+\frac{1}{3p}$ To combine each term we need a least common denominator, here lcd = $6p$

$$\Rightarrow \frac{7}{p}\cdot\frac{6}{6}+\frac{9}{2p}\cdot\frac{3}{3}+\frac{1}{3p}\cdot\frac{2}{2}=\frac{42}{6p}+\frac{27}{6p}+\frac{2}{6p}=\frac{71}{6p}$$

e.g. $\frac{x+1}{x^2+5x+6}-\frac{5x-1}{x^2-x-12} = \frac{x+1}{(x+3)(x+2)}-\frac{5x-1}{(x+3)(x-4)}$

$$=\frac{x+1}{(x+3)(x+2)}\cdot\frac{(x-4)}{(x-4)}-\frac{5x-1}{(x+3)(x-4)}\cdot\frac{x+2}{x+2}$$

$$=\frac{(x+1)(x-4)-(5x-1)(x+2)}{(x+2)(x+3)(x-4)}=\frac{(x^2-3x-4)-(5x^2+9x-2)}{(x+2)(x+3)(x-4)}$$

$$=\frac{-4x^2-12x-2}{(x+2)(x+3)(x-4)}=\frac{-2(2x^2+6x+1)}{(x+2)(x+3)(x-4)}$$

§R.4 : Equations

<u>Linear Equations</u> : eqn's of the form $ax+b=0$ with $a\neq 0$

<u>Properties of Equality</u> : $\forall\, a,b,c \in \mathbb{R}$

(1) $a=b \Rightarrow a+c=b+c$ (Addition)

(2) $a=b \Rightarrow ac=bc$ (Multiplication)

e.g. $2x-5+8 = 3x+2(2-3x)$

$\Leftrightarrow 2x+3 = 3x+4-6x \Leftrightarrow 2x+3 = 4-3x$

$\Leftrightarrow 2x+3x = 4-3$

$\Leftrightarrow 5x=1 \Leftrightarrow \boxed{x=\frac{1}{5}}$

<u>Quadratic Eqn</u> : Equations whose leading term has power 2 and are of the form $ax^2+bx+c=0$ with $a\neq 0$
This is called standard form

<u>Zero Factor Property</u> : $\forall\, a,b \in \mathbb{R}$, $ab=0 \Rightarrow a=0, b=0$ or both

either factor the equation and use the zero factor property or use the <u>quadratic formula</u>

$\boxed{\text{Quadratic Formula}}$

The solutions of the quadratic equation $ax^2+bx+c=0$, $a\neq 0$ are given by:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

e.g. Solve $x^2 - 4x = 5$

1. Rewrite in Standard form: $x^2 - 4x - 5 = 0$

Factor Approach: $x^2 - 4x - 5 = (x+1)(x-5) = 0 \Rightarrow x = 5$ or $x = -1$
(Zero factors property)

Quadratic Formula: Here $a = 1$, $b = -4$, $c = -5$

$$\Rightarrow x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4(1)(-5)}}{2(1)} = \frac{4 \pm \sqrt{16+20}}{2} = \frac{4 \pm \sqrt{36}}{2}$$

$$= \frac{4 \pm 6}{2} = 2 \pm 3 \Rightarrow \boxed{x_1 = 5,\ x_2 = -1}$$

Perfect to have radicals in your answer however if there is a negative number under the radicand (i.e. $3 + \sqrt{-5}$) then this is an example of a complex #

Sometimes you can solve quadratics directly:

e.g. $\frac{3}{x^2} - 12 = 0 \Leftrightarrow \frac{3}{x^2} = 12 \Leftrightarrow 12x^2 = 3$

$$\Leftrightarrow x^2 = \frac{3}{12} = \frac{1}{4} \Rightarrow x = \pm\sqrt{\frac{1}{4}} = \boxed{\pm\frac{1}{2}}$$

It is possible to obtain a solution to a rational equation that makes the denominator $= 0$, which is NOT allowed, called Extraneous Solution

(Solve) e.g. $\frac{2}{x-3} + \frac{1}{x} = \frac{6}{x(x-3)}$

$$\Rightarrow \frac{2x + x - 3}{(x-3)(x)} = \frac{6}{x(x-3)} \Rightarrow 3x - 3 = 6 \Rightarrow 3x = 9 \Rightarrow x = 3$$

However plugging $x = 3$ into the original equation makes 2 denominators 0, so we say No solution or $\emptyset$ for this equation.

§R.5 : Inequalities

Inequality Symbols : $\forall a,b,c \in \mathbb{R}$

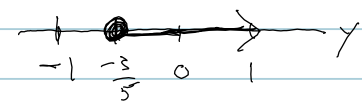
1. if $a<b$ then $a+c<b+c$
2. if $a<b$ and $c>0$, then $ac<bc$
3. if $a<b$ and $c<0$, then $ac>bc$ (sign flip! e.g.) $2<3$ but $-2>-3$)

e.g.) Linear Inequality: Solve

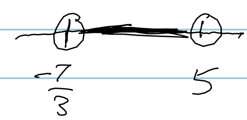
sign flip!

$$4-3y \le 7+2y \iff \frac{-5y}{-5} \le \frac{3}{-5} \iff y \ge -\frac{3}{5}$$

You can graph the solution on a # line:



e.g.) $-\frac{7}{3} < m < 5$



Since here we don't include endpoints, they are displayed as open circles o

Interval Notation

$a \le x \le b \equiv x \in [a,b] \equiv$ "x is an element of the closed interval $[a,b]$"

$a<x<b \equiv x \in (a,b) \equiv$ "x is in the open interval on (a,b)"

Intervals can also be half open and half closed

e.g.) Quadratic Inequalities: Solve $x^2-x<12$

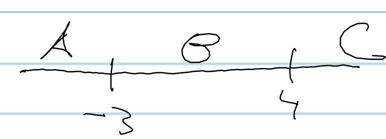
Rewrite the inequality as : $x^2-x-12<0$

Now solve $x^2-x-12=0$

Factor $(x-4)(x+3)=0 \Rightarrow x=4$ or $x=-3$

Now at each of these points there can be a sign change, so we must consider these 3 intervals

$(-\infty,-3]$, $[-3,4]$, $[4,\infty)$
A B C



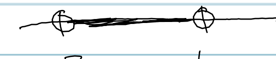
Now ch test points from each of these intervals and plug into $x^2-x-12<0$

$A=[-\infty,-3)$ ~~choose~~ choose -4 $\Rightarrow (-4)^2-(-4)-12=8>0$ ✗ Not Satisfied

$B=[-3,4]$ choose $0 \Rightarrow (0)^2-(0)-12=-12<0$ ✓ Satisfied

$C=[4,\infty)$ choose $5 \Rightarrow (5)^2-(5)-12=8>0$ ✗ NOT Satisfied

So the solution is given by the interval $(-3,4)$



e.g. Rational Inequality: Solve $\frac{2x-3}{x}\geq 1$

First we want to see when this expression is equal to the right hand side (1) since this will tell us the critical point:

$$\frac{2x-3}{x}=1 \iff 2x-3=x \iff \boxed{x=3}$$

Now there can also be a sign change when the denominator changes from negative to positive. Thus we must consider when the denominator equals 0 as another critical point $\boxed{x=0}$

That gives us 3 intervals to check: $(-\infty,0]$, $[0,3]$, $[3,\infty)$

$[-\infty,0]$ choose $-1 \Rightarrow \frac{2(-1)-3}{-1}=5\geq 1$ ✓

Check endpoint: $0 \Rightarrow \frac{2(0)-3}{0}=$ ∅ ✗

$[0,3]$ choose $1 \Rightarrow \frac{2(1)-3}{1}=\frac{-1}{1}=-1\ngeq 1$ ✗

check endpoint

$3 \Rightarrow \frac{2(3)-3}{3}=+3\geq 1$ ✓

$[3,\infty)$ choose $4 \Rightarrow \frac{2(4)-3}{4}=\frac{5}{4}\geq 1$ ✓

The solution is given by

$$\Rightarrow \boxed{(-\infty,0)\cup[3,\infty)}$$

§R.6 Exponents

Definition: if n is a natural #, $n \in \mathbb{N} = \{1, 2, \ldots\}$, then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}} \; ; \; a^0 = 1 \; ; \; a^{-n} = \frac{1}{a^n}$$

Properties: For $m, n \in \mathbb{Z}$, $a, b \in \mathbb{R}$

1. $a^m \cdot a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $(ab)^m = a^m \cdot b^m$
5. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$
6. $(-a)^n = \begin{cases} a^n & \text{if } n \text{ is even integer } (2\mathbb{Z}) \\ -a^n & \text{if } n \text{ is an odd integer} \end{cases}$

e.g.) Simplifying Exponential expressions:

(1) $3x^2 \cdot x^4 = 3x^6$

(2) $(2m^3)^4 = 2^4 m^{4 \cdot 3} = 16m^{12}$

(3) $\left(\frac{x^2}{y^3}\right)^6 = \frac{x^{6 \cdot 2}}{y^{6 \cdot 3}} = \frac{x^{12}}{y^{18}}$

(4) $\frac{a^{-3} b^5}{a^4 b^{-7}} = \frac{b^{5+7}}{a^{4+3}} = \frac{b^{12}}{a^7}$

(5) $$\frac{x^{-2} - y^{-2}}{x^{-1} - y^{-1}} = \frac{\frac{1}{x^2} - \frac{1}{y^2}}{\frac{1}{x} - \frac{1}{y}} = \frac{\frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2}}{\frac{y}{xy} - \frac{x}{xy}} = \frac{\frac{y^2 - x^2}{x^2y^2}}{\frac{y-x}{xy}} = \frac{(y+x)\cancel{(y-x)}}{x^{\cancel{2}}y^{\cancel{2}}} \cdot \frac{\cancel{xy}}{\cancel{(y-x)}}$$

$$= \boxed{\frac{y+x}{xy}}$$

Exponents can also be rational numbers, or roots.
The most familiar is: $\sqrt{x} = x^{1/2}$ which can generalize to:

$$\boxed{a^{m/n} = (a^{1/n})^m, \; \forall a \in \mathbb{R}, \; m, n \in \mathbb{Z}}$$

e.g.) $27^{2/3} = (27^{1/3})^2 = ((3^3)^{1/3})^2 = 3^2 = 9$

§R.7 : Radicals

Instead of exponents we can also use radical notation:

Radicals: If n is an even natural # and $a \geq 0$ or n is an odd natural # then

$\boxed{a^{1/n} = \sqrt[n]{a}}$; $\boxed{a^{m/n} = \left(\sqrt[n]{a}\right)^m = a^{m/n} = \sqrt[n]{a^m}}$

Properties: $\forall a, b \in \mathbb{R}$, $m, n \in \mathbb{N}$ st $\sqrt[n]{a}, \sqrt[n]{b} \in \mathbb{R}$

1. $\left(\sqrt[n]{a}\right)^n = a$

2. $\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ even} \\ a & \text{if } n \text{ odd} \end{cases}$

3. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

4. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$ $(b \neq 0)$

5. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

note property 2 due to the fact: $(-a)^n = -a^n$ if n is odd and
$(-a)^n = a^n$ if n is even

Def^n : The absolute value of x is defined as $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
(i.e. always positive!)

Note : $\sqrt[n]{a^n + b^n} \neq \sqrt[n]{a^n} + \sqrt[n]{b^n}$

Sometimes its useful to remove the radical from either the numerator or denominator of an expression, we call this rationalization

E.g. (1) $\dfrac{4}{\sqrt{3}}$ To rationalize the denominator, simply multiply by $\dfrac{\sqrt{3}}{\sqrt{3}} \Rightarrow \dfrac{4}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{4\sqrt{3}}{(\sqrt{3})^2} = \boxed{\dfrac{4\sqrt{3}}{3}}$

(2) $\dfrac{1}{1-\sqrt{2}} = \dfrac{1}{1-\sqrt{2}} \cdot \dfrac{(1+\sqrt{2})}{(1+\sqrt{2})} = \dfrac{1+\sqrt{2}}{1+(\sqrt{2}-\sqrt{2})-2} = \dfrac{1+\sqrt{2}}{-1} = \boxed{-1-\sqrt{2}}$

(3) $\dfrac{2}{\sqrt[3]{x}} = \dfrac{2}{\sqrt[3]{x}} \cdot \dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}} = \dfrac{2\sqrt[3]{x^2}}{\sqrt[3]{x^3}} = \boxed{\dfrac{2\sqrt[3]{x^2}}{x}}$