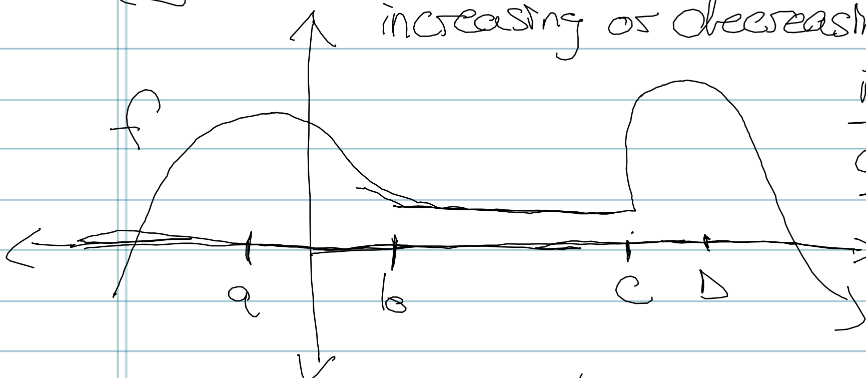
§5.1: Increasing and Decreasing Functions

Def^n: Let f be a function defined on some interval.
Let x_1 and x_2 be two numbers in the interval.

(1) f is <u>increasing</u> on the interval if
$f(x_1) < f(x_2)$ whenever $x_1 < x_2$

(2) f is <u>decreasing</u> on the interval if
$f(x_1) > f(x_2)$ whenever $x_1 < x_2$

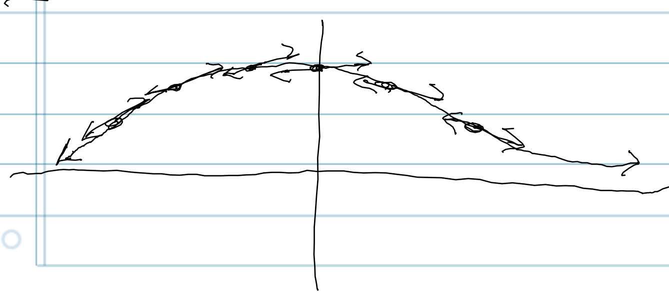
e.g.) We can see this graphically when some function, f, is increasing or decreasing:



<u>increasing</u>: $(-\infty, a)$, (c, d)
<u>decreasing</u>: (a, b), (d, ∞)
<u>Constant</u>: (b, c)

<u>Test for increasing/decreasing intervals for a function</u>
- Suppose the derivative of f exists at each point in an open interval:
- if $f'(x) > 0$, for each x in the interval, f is increasing
- if $f'(x) < 0$, for each " " " " f is decreasing
- if $f'(x) = 0$, " " " " " " f is constant

e.g.) Consider the derivative (slope of the tangent line) of the function:



How can we find the intervals where a derivative is positive or negative?

Critical Numbers

The critical numbers for a function, f, are those numbers, c, in the domain of f for which

$$f'(c) = 0 \quad \text{or} \quad f'(c) \text{ DNE (i.e. is undefined)}$$

- A critical point is a point whose x-coordinate is a critical number and has $y = f(c)$. (ie the point $(c, f(c))$)

Applying the Test

(1.) locate the critical numbers of f along the number line (i.e. solve $f'(x) = 0$ and $f'(x)$ DNE)

(2.) Choose test points between critical numbers and evaluate f' at each. If $f'(x_{test}) > 0$ then f is increasing on the interval. If $f'(x_{test}) < 0$ then f is decreasing

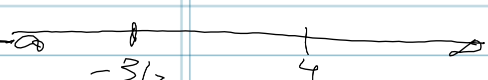
e.g. Determine the intervals on which f is increasing and decreasing.

(A) $f(x) = 4x^3 - 15x^2 - 72x + 5$; First find the derivative

$$f'(x) = 12x^2 - 30x - 72 = 6(2x^2 - 5x - 12)$$

Is $f'(x) = 0$ possible? Yes!

$$f'(x) = 6(2x^2 - 5x - 12) = 6(2x+3)(x-4) = 0$$



$\Rightarrow$ $2x+3=0$ or $x-4=0$

$\Rightarrow \boxed{x = -3/2}$ $\Rightarrow \boxed{x = 4}$ are critical numbers

This gives us 3 intervals:

$(-\infty, -3/2)$ $(-3/2, 4)$ $(4, \infty)$

Now we need to test points in these intervals to see if the function is increasing or decreasing along the interval.

$(-\infty, -3/2)$

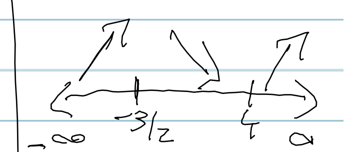
$x_{T_1} = -2 \Rightarrow f'(-2) = 6((-2)(2)+3)((-2)-4) = 6(-1)(-6) = 36 > 0$

so increasing on $(-\infty, -3/2)$

$(-3/2, 4)$

$x_{T_2} = 0$ $f'(0) = 6(0+3)(0-4) = 6(3)(-4) = -72 < 0$

$\Rightarrow f$ is decreasing on $(-3/2, 4)$



$(4, \infty)$

$x_{T_3} = 5 \Rightarrow f'(5) = 6(5 \cdot 2+3)(5-4) = 6(13)(1) > 0$

$\Rightarrow f$ increasing on $(4, \infty)$

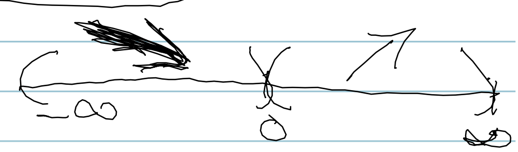
(8) $f(x) = \sqrt{x^2+1} = (x^2+1)^{1/2} \Rightarrow f'(x) = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}$

Is the function undefined anywhere? Nope

Can the function equal 0?

$\frac{x}{\sqrt{x^2+1}} = 0 \Rightarrow \boxed{x = 0, \text{ critical Number}}$

This gives us 2 intervals to check



$(-\infty, 0)$

choose $x_{T_1} = -1 \Rightarrow f'(-1) = \frac{-1}{\sqrt{(-1)^2+1}} < 0$ so decreasing

$(0, \infty)$

choose $x_{T_2} = 1 \Rightarrow f'(1) = \frac{1}{\sqrt{1^2+1}} > 0$ so increasing

(C) $f(x)=(x+1)^{4/5} \Rightarrow f'(x)=\frac{4}{5}(x+1)^{-1/5}(1)=\frac{4}{5(x+1)^{1/5}}$

Can the function be 0? nope

Is the function ever undefined? $(x+1)^{1/5}=0 \Rightarrow x=-1$

(Since the function is undefined here this is NOT a critical number

however we proceed in the same manner)

Gives us 2 intervals

$(-\infty,-1)$

$x_{T_1}=-2 \Rightarrow f'(-2)=\frac{4}{5(-2+1)^{1/5}}=\frac{4}{5(-1)^{1/5}}<0$, decreasing

$(-1,\infty)$

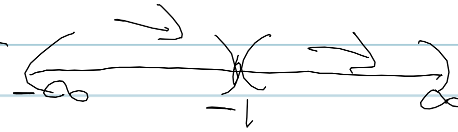
$x_{T_2}=0 \Rightarrow f'(0)=\frac{4}{5(0+1)^{1/5}}=\frac{4}{5}>0$, increasing

(D) $f(x)=\frac{x+2}{x+1} \Rightarrow f'(x)=\frac{(x+1)(1)-(x+2)(1)}{(x+1)^2}=\frac{x+1-x-2}{(x+1)^2}=\frac{-1}{(x+1)^2}$

Can $f'(x)=0$? Nope

Can $f'(x)=$ DNE Yes! $(x+1)^2=0 \Rightarrow x=-1$ (again Not a critical number since $f'(-1)=$ DNE)

Gives us 2 intervals to check



$(-\infty,-1)$

$x_{T_1}=-2 \Rightarrow f'(-2)=\frac{-1}{(-2+1)^2}=\frac{-1}{1^2}=-1<0$ decreasing

$(-1,\infty)$

$x_{T_2}=0 \Rightarrow f'(0)=\frac{-1}{(0+1)^2}=-1<0$ decreasing

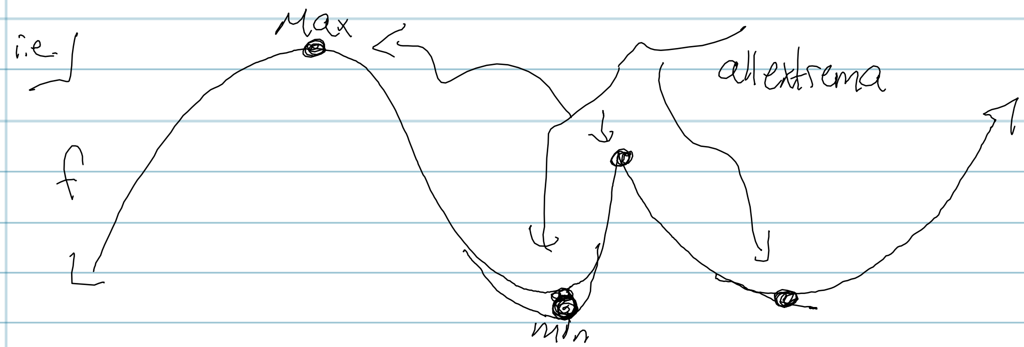
So function is always decreasing.

§5.1: Relative Extrema

Def^n: A function has a relative (or local) extremum (plural: extrema) at c if it has either a relative maximum or minimum, these

If c is an endpoint of the domain of f we only consider x in the half-open interval that is in the domain

i.e.



Relative Maximum or Minimum

- Let c be a number in the domain of f, then $f(c)$ is a relative (or local) maximum for f is $\exists\,(a,b)$ an open interval containing c ST

$$f(x) < f(c) \quad ; \quad \forall x \in (a,b)$$

- Likewise $f(c)$ is a local minimum for f if $\exists\,(a,b)$ containing c ST $f(x) > f(c)$; $\forall x \in (a,b)$

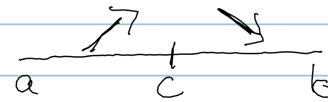
- If a function f has a relative extremum at c, then c is a critical number, or c is an endpoint of the domain

First Derivative Test

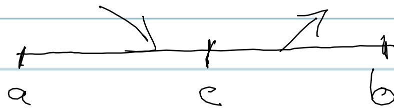
- Let c be a critical number for a function f. Suppose that f is continuous on (a,b) and differentiable on (a,b), except possibly at c, and that c is the only critical number for f in (a,b)

Then the following are true:

1. $f(c)$ is a relative maximum of f if $f'(x) \geq 0$ on (a,c) and $f'(x) < 0$ on (c,b)



2. $f(c)$ is a relative minimum of f if $f'(x) < 0$ when x is in (a,c) and $f'(x) > 0$ in (c,b)



e.g. Find the values of any relative extrema:

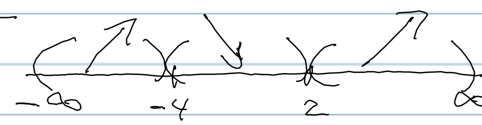
(A) $f(x) = x^3 + 3x^2 - 24x + 2$. First find the derivative!

$$f'(x) = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x+4)(x-2)$$

Now $f'(x) = 0 \Rightarrow x+4=0$ or $x-2=0$

$\Rightarrow x_c = -4$ or $x_c = 2$

This gives us some intervals to check:



Test Points

$(-\infty, -4)$

$x_T = -5 \Rightarrow f'(-5) = 3(-1)(-7) > 0$ (increasing)

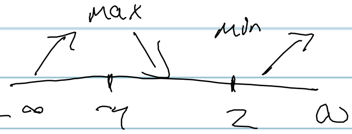
$(-4, 2)$

$x_T = 0 \Rightarrow f'(0) = 3(4)(-2) < 0$ (decreasing)

$(2, \infty)$

$x_T = 3 \Rightarrow f'(3) = 3(7)(+1) > 0$ (increasing)

So we have



Now we have to evaluate at each of these critical points to find the location of the minimum, maximum

Max @ $x = -4 \Rightarrow f(-4) = (-4)^3 + 3(-4)^2 - 24(-4) + 2 = \boxed{82}$

$\Rightarrow (-4, 82)$

Min @ $x = 2$

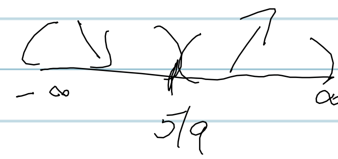
$\Rightarrow f(2) = 2^3 + 3 \cdot 2^2 - 24 \cdot 2 + 2 = \boxed{-26}$

(8) $f(x) = \dfrac{(5-9x)^{2/3}}{7} + 1$ Find the derivative?

$$f'(x) = \frac{2}{\cancel{3}} \cdot \frac{1}{7} (5-9x)^{-1/3} (\cancel{-9}^{\,-3}) = \frac{-6}{7(5-9x)^{1/3}}$$

Can $f'(x) = 0$? NO! Can $f'(x)$ be undefined? Yes!

$$(5-9x)^{1/3} = 0 \Rightarrow 5-9x = 0 \Rightarrow x = 5/9$$



Test Pts

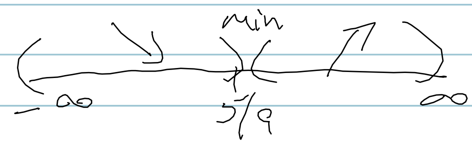
$(-\infty, 5/9)$

$x_T = 0 \Rightarrow f'(0) = \dfrac{-6}{7(5)^{1/3}} < 0$ (decreasing ↘)

$(5/9, \infty)$

$x_T = 1 \Rightarrow f'(1) = \dfrac{-6}{7(5-9)^{1/3}} = \dfrac{-6}{7(-4)^{1/3}} > 0$ (increasing ↗)

So we have:



So we have a minimum at $x = 5/9$, plug it in to get the value:

$$f(5/9) = \frac{(5-9\cdot\frac{5}{9})^{2/3}}{7} + 1 = 0 + 1 = 1$$

$\Rightarrow$ Minimum at $(5/9, 1)$

§5.3 : Higher Derivatives, Concavity, and the Second derivative test

Def^n: If a function, f, has a derivative, f', then the derivative of f', if it exists, is the second derivative of f denoted f'':

$$D_x[f'(x)] = f''(x) \equiv \text{The second derivative of } f$$

Notation for higher derivatives:

The second derivative of $y = f(x)$ can be written as

$$f''(x) \; ; \; \frac{d^2y}{dx^2} \; ; \; \text{or } D_x^2[f(x)] \text{ or } \ddot{y} \text{ or } \frac{d^2}{dx^2}[f(x)]$$

For the 3rd derivative we may write $f'''(x)$, but when $n > 3$ instead of prime notation: $f^{(n)}(x) = \frac{d^ny}{dx^n} = D_x^n[f(x)]$

e.g.) find the second and third derivative:

(A) $f(x) = 4x^3 + 5x^2 + 6x - 7$

$$\Rightarrow f'(x) = 12x^2 + 10x + 6 \Rightarrow f''(x) = 24x + 10 \Rightarrow f'''(x) = 24$$

(B) $f(x) = (x^3 - 1)^2 \Rightarrow f'(x) = 2(x^3-1)^1(3x^2) = 6x^5 - 6x^2$

$$\Rightarrow f''(x) = 30x^4 - 12x \Rightarrow f'''(x) = 120x^3 - 12$$

(C) $f(x) = \frac{\ln x}{e^x} = \underset{u}{\ln x}\,(\underset{v}{e^{-x}})$ Find $f'(x)$ and $f''(x)$

$$\Rightarrow f'(x) = uv' + u'v = (\ln x)(-e^{-x}) + \frac{1}{x}e^{-x} = -\underset{u_1}{\ln x}\,\underset{v_1}{e^{-x}} + \underset{u_2}{x^{-1}}\,\underset{v_2}{e^{-x}}$$

$$f''(x) = u_1v_1' + u_1'v_1 + u_2v_2' + u_2'v_2 = (-\ln x)(-e^{-x}) + (-x^{-1})e^{-x} + x^{-1}(-e^{-x}) + (-x^{-2})e^{-x}$$

(D) Suppose the position of plane relative to the airport is $s(t) = 4t^3 + 3t^2 + 2$

Find velocity and acceleration

$v(t) = s'(t) = 12t^2 + 6t$

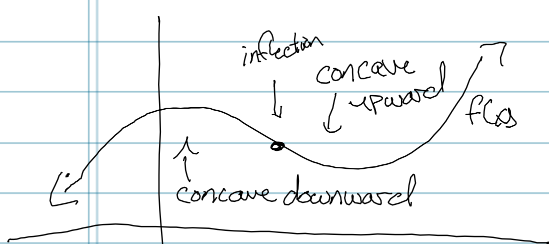
$a(t) = s''(t) = v'(t) = 24t$

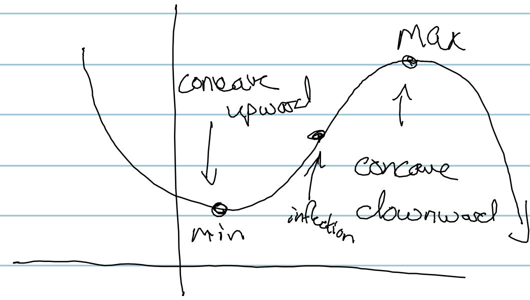
- The first derivative tells us when a function is increasing or decreasing (hence extrema)
- The second derivative gives us the rate of change of the first derivative (i.e. the rate of change of the velocity is the acceleration")

So $f''(x) \geq 0 \Rightarrow$ speeding up

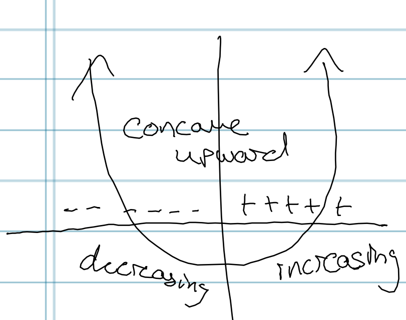
$f''(x) < 0 \Rightarrow$ slowing down

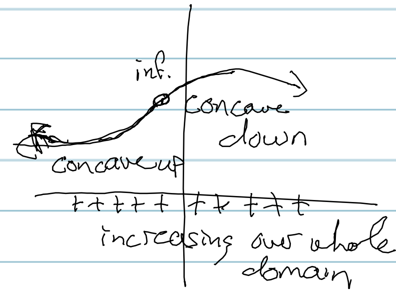
In the context of a graph, $f''(x)$ tells us about the shape or "inflection"

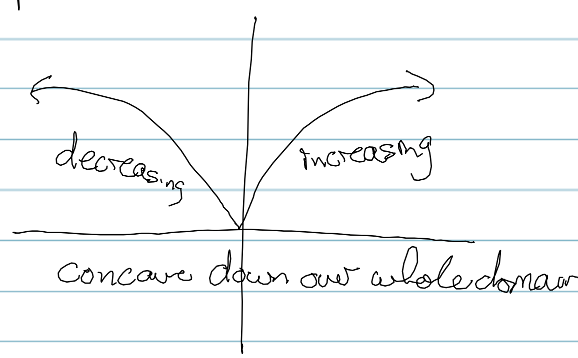




- Concave downward "spills water"
- concave upward "holds water"
- A point where the concavity of the graph of a function changes is known as an inflection point







Test for concavity

Let f be a function with derivatives f' and f'' existing at all points in the interval (a,b). Then f is concave upward on (a,b) if $f''(x) > 0$ $\forall x \in (a,b)$ and concave downward on (a,b) if $f''(x) < 0$, $\forall x \in (a,b)$

- An inflection point (occurs at x_i) for a function f occurs when $f''(x_i) = 0$ or $f''(x_i)$ DNE

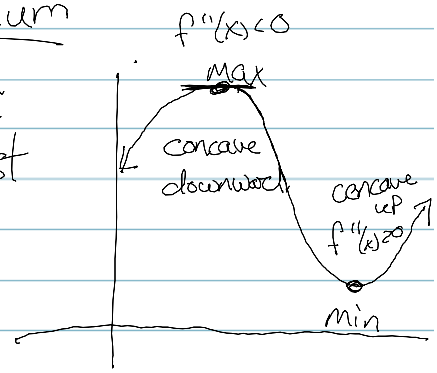
Second derivative Test (for extrema)

Let $f''(x)$ exist on some open interval containing c (except possibly c itself) and let $f'(c) = 0$

(1) If $f''(c) > 0$ then $f(c)$ is a <u>relative minimum</u>

(2) If $f''(c) < 0$ then $f(c)$ is a <u>relative maximum</u>

(3) If $f''(c) = 0$ or $f''(c)$ DNE then the test gives No information about extrema (so must use first derivative test)



e.g.) Find all intervals where the function is concave upward or downward, find all inflection points and relative extrema using the second derivative test.

(A) $f(x) = -x^3 - 12x^2 - 45x + 2$

First find critical pts:

$$f'(x) = -3x^2 - 24x - 45 = -3(x^2 + 8x + 15) = -3(x+5)(x+3) = 0$$

$$\Rightarrow x_c = -3, -5$$

$f''(x) = -6x - 24$ Now plug in to get extrema:

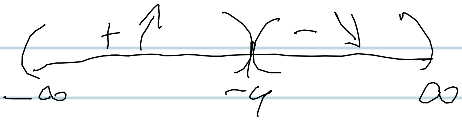
$$f''(-3) = -6(-3) - 24 = 18 - 24 < 0 \Rightarrow \text{max @ } (-3, f(-3))$$

$$f''(-5) = -6(-5) - 24 = 30 - 24 > 0 \Rightarrow \text{min @ } (-5, f(-5))$$

Now we need to find concavity, set $f''(x) = 0$

$$f''(x) = 0 \Rightarrow -6x - 24 = 0 \Rightarrow x = \frac{24}{-6} = -4$$

This gives us intervals to check



$(-\infty, -4)$

$x_1 = -5 \Rightarrow f''(-5) = 30 - 24 > 0$ (concave ↗)

$(-4, \infty)$

$x_1 = 0 \Rightarrow f''(0) = 0 - 24 < 0$ (concave down)

Since concavity switches we have an inflection point

@ $(-4, f(-4)) = (-4, 54)$

(8) $f(x) = 2e^{-x^2}$

$$f'(x) = 2e^{-x^2}(-2x) = -4xe^{-x^2}$$

Critical Pts

$$f'(x) = 0 \Rightarrow -4x\left(e^{-x^2}\right) = 0 \Rightarrow \boxed{x_c = 0}$$

$$f''(x) = -4\left[x\left(e^{-x^2} \cdot -2x\right) + 1\left(e^{-x^2}\right)\right] = -4e^{-x^2}\left[1 - 2x^2\right]$$

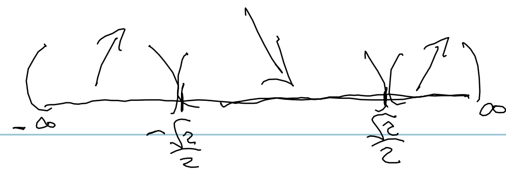
Plug in x_c to classify extremum:

$$f''(0) = -4e^0[1-0] = -4 < 0 \Rightarrow \text{Max @ } (0, f(0)) = (0, 2)$$

Now check for concavity

$$f''(x) = -4e^{-x^2}\left[1 - 2x^2\right] = 0 \Rightarrow 1 - 2x^2 = 0 = (1 + \sqrt{2}x)(1 - \sqrt{2}x)$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx .707$$

This gives us intervals to check 

$(-\infty, -\sqrt{2}/2)$

$x_T = -1 \Rightarrow f''(-1) = -4e^{(-1)^2}[-2(-1)^2+1] = -4e^{1^2}[-1] > 0$ <u>concave up</u>

$(-\sqrt{2}/2, \sqrt{2}/2)$

$x_T = 0 \Rightarrow f''(0) = -4e^0[-2\cdot 0+1] < 0$ <u>concave down</u>

$(\frac{\sqrt{2}}{2}, \infty)$

$x_T = \underline{1} \Rightarrow f''(1) = -4e^{1^2}[-2(1)^2+1] > 0$ <u>concave up</u>

So concavity changed twice $\Rightarrow$ 2 inflection points

$x = -\frac{\sqrt{2}}{2} \Rightarrow f(-\frac{\sqrt{2}}{2}) = 2e^{-(-\frac{\sqrt{2}}{2})^2} = 2e^{-\frac{2}{4}} = 2/\sqrt{e}$

$x = \frac{\sqrt{2}}{2} \Rightarrow f(\sqrt{2}/2) = 2e^{-(\frac{\sqrt{2}}{2})^2} = 2/\sqrt{e}$

So 2 inflection pts @ $(\pm\frac{\sqrt{2}}{2}, 2/\sqrt{e})$