§4.2 : Derivatives of Products and Quotients

We know that the derivative of the sum is the sum of the derivatives. What about the derivative of a product or quotient?

Is it just $\frac{d}{dx}[f(x)\cdot g(x)] \overset{?}{=} f'(x)\cdot g'(x)$? Let's investigate.

Suppose $f(x) = 3x+2$ and $g(x) = 4x^3$

Then (power rule) $\frac{d}{dx}(f(x)) = 3$ and $\frac{d}{dx} g(x) = 12x^2$ but:

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(12x^4 + 8x^3) = 48x^3 + 24x^2 \neq 3\cdot 12x^2 = f'(x)g'(x)$$

So, we must devise a formula for the derivative of a product!

For some general function $f(x) = u(x)v(x)$, lets evaluate $f'(x)$ using the formal definition of the derivative:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

neaky trick: dd and subtract (x+h)v(x) from he numerator

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) + [u(x+h)v(x) - u(x+h)v(x)] - u(x)v(x)}{h}$$

rearrange, factor

$$= \lim_{h\to 0} \frac{u(x+h)[v(x+h) - v(x)]}{h} + \lim_{h\to 0} \frac{v(x\cancel{+h})[u(x+h) - u(x)]}{h}$$

$$= u(x)v'(x) + v(x)u'(x)$$

This gives us a formula for the derivative of the product, AKA The Product Rule!

The Product Rule: Let $f(x) = u(x)v(x)$ st u and v are differentiabl (i.e. $u'(x)$, $v'(x)$ exists). Then

$$\boxed{f'(x) = u(x)v'(x) + v(x)u'(x)}$$

e.g.) Find the derivative with the Product rule

(A) $f(x) = \underbrace{(3x+2)}_{u}\underbrace{(4x^3)}_{v} \Rightarrow f'(x) = \underbrace{(3x+2)}_{u}\underbrace{(3\cdot 4x^{3-1})}_{v'} + \underbrace{(3x^{1-1})}_{u'}\underbrace{(4x^3)}_{v} = (3x+2)(12x^2) + 12x^3$

$= 36x^3 + 12x^3 + 24x^2$

$= \boxed{48x^3 + 24x^2}$

(B) $y = (x+1)(\sqrt{x}+2) \Rightarrow \frac{dy}{dx} = (x+1)\left[\frac{1}{2}x^{\frac{1}{2}-1} + 0\right] + [1+0](\sqrt{x}+2) = (x+1)\left(\frac{1}{2}x^{-1/2}\right) + (\sqrt{x}+2)$

$= (x+1)(x^{1/2}+2)$

$= \frac{1}{2}x^{1-\frac{1}{2}} + \frac{1}{2}x^{-1/2} + \sqrt{x} + 2 = \boxed{\frac{3}{2}x^{+1/2} + \frac{1}{2}x^{-1/2} + 2}$

Combine

$= \frac{3}{2}\sqrt{x} + 2 + \frac{1}{2\sqrt{x}}$

(C) $p(y) = \left(\frac{1}{y} + \frac{1}{y^2}\right)\left(\frac{2}{y^3} - \frac{5}{y^4}\right) \Rightarrow p'(y) = (y^{-1}+y^{-2})(-6y^{-4}+20y^{-5}) + (-y^{-2}-2y^{-3})(2y^{-3}-5y^{-4})$

$= (y^{-1}+y^{-2})(2y^{-3}-5y^{-4})$

$= -6y^{-5} + 20y^{-6} - 6y^{-6} + 20y^{-7}$

$-2y^{-5} + 5y^{-6} - 4y^{-6} + 10y^{-7}$

$5y^{-4}$

rearrange, combine

$\Rightarrow \boxed{p'(y) = -8y^{-5} + 15y^{-6} + 30y^{-7}}$

How about the derivative of the quotient of functions?
For that, we have:

The Quotient Rule:

If $f(x) = \frac{u(x)}{v(x)}$ and u', v' exist with $v(x) \neq 0$, then:

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Or, you can remember the jingle ♫

Low D High − High D Low all over Low Low

$$\left\{\frac{(v(x))(u'(x)) - u(x)v'(x)}{v(x)v(x)}\right\}$$

e.g.) Find the derivative with the quotient rule

(A) $f(t) = \dfrac{4t^2+11}{t^2+3}$

Here we can use the quotient rule with: $u(x) = 4t^2+11$, $v(x) = t^2+3$

By Power Rule: $u'(x) = 8t$; $v'(x) = 2t$

Plug into our quotient rule formula:

$$f'(t) = \frac{u'v - uv'}{v^2} = \frac{(8t)(t^2+3) - [(4t^2+11)(2t)]}{(t^2+3)^2} = \frac{\cancel{8t^3} + 24t - \cancel{8t^3} - 22t}{(t^2+3)^2}$$

$$= \boxed{2t/(t^2+3)^2}$$

* Notice, in general you can usually leave the denominator in factored form

(B) $f(x) = \dfrac{(3x^2+1)(2x-1)}{5x+4}$

Here let $u(x) = (3x^2+1)(2x-1)$ and $v(x) = 5x+4$, $v'(x) = 5$

To find $u'(x)$, use the product rule

$$u'(x) = (3x^2+1)[2] + [6x](2x-1) = 6x^2 + 2 + 12x^2 - 6x$$
$$= 18x^2 - 6x + 2$$

Now, use the quotient rule!

$$f'(x) = \frac{(18x^2-6x+2)(5x+4) - (3x^2+1)(2x-1)(5)}{[5x+4]^2}$$

$$= \frac{90x^3 + 72x^2 - 30x^2 + \overset{(-24x+10x)}{8} - [(6x^3 - 3x^2 + 2x - 1)5]}{(5x+4)^2}$$

$$= \frac{90x^3 + 72x^2 - 30x^2 + \overset{(-24x+10x)}{8} - 30x^3 + 15x^2 - 10x + 5}{(5x+4)^2}$$

$$= \frac{60x^3 + 57x^2 - 24x + 13}{(5x+4)^2}$$

Average Cost: Suppose $y = C(x)$ gives the total cost to manufacture x items

Then the average cost per item is $\bar{y} = \bar{C}(x) = \frac{C(x)}{x}$

i.e. The cost of x items over the total # of items

Marginal Average Cost

This is simply the derivative (i.e. rate of change) of the average cost function

So if the average cost is given by: $\bar{C}(x) = C(x)/x$, then

$$\text{Marginal Avg Cost} \equiv \bar{C}'(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right)$$

A company may want to make the average cost as small as possible:

e.g.) Suppose the total ~~average~~ cost (in 100's of dollars) to produce x units of skis is $C(x) = \frac{3x+2}{x+4}$; Find the marginal average cost function.

Before we can find the marginal average cost, we ~~k~~ need the average cost function.

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{3x+2}{x+4} \cdot \frac{1}{x} = \frac{3x+2}{x^2+4x}$$

To find the marginal average cost we need to use the quotient rule:

$$\bar{C}'(x) = \frac{d}{dx}\left(\frac{C(x)}{x}\right) = \frac{(3)(x^2+4x) - (3x+2)(2x+4)}{(x^2+4x)^2}$$

$$= \frac{3x^2 + \cancel{12x} - [6x^2 + \cancel{12x} + 4x + 8]}{(x^2+4x)^2} = \frac{-3x^2 - 4x - 8}{(x^2+4x)^2}$$

One may want to minimize the average cost. This generally occurs when the marginal average cost (its derivative ?) is 0. This "minimization" technique will be covered in more detail in the next chapter.

§1.3: The Chain Rule

The chain rule describes how one takes the derivative of a composition of functions. Lets review:

Composite Function

Let f and g be functions. The composite function, $g \circ f(x)$ is the function whose values are given by $g(f(x))$, $\forall x$ in the domain of f ST $f(x)$ is in the domain of g

e.g.) Let $f(x) = 2x-1$ and $g(x) = \sqrt{3x+5}$. Find:

(A) $g \circ f(x) = g(f(x)) = \sqrt{3(2x-1)+5} = \sqrt{6x-3+5} = \sqrt{6x+2}$

(B) $f \circ g(x) = 2\left[\sqrt{3x+5}\right] - 1$

(C) $g \circ f(4)$:

$f(4) = 2 \cdot 4 - 1 = 7$; $g(7) = \sqrt{3 \cdot 7 + 5} = \sqrt{21+5} = \sqrt{26}$

OR

$g \circ f(4) = \sqrt{3[2 \cdot 4 - 1]+5} = \sqrt{3[7]+5} = \sqrt{21+5} = \sqrt{26}$

e.g.) Write each as the composition of 2 functions f and g so that $h(x) = f \circ g(x)$

(A) $h(x) = 2(4x+1)^2 + 5(4x+1)$

Let $g(x) = 4x+1 \Rightarrow h(x) = 2[g(x)]^2 + 5[g(x)]$

Now let $f(x) = 2x^2 + 5x$

$\Rightarrow h(x) = f \circ g(x) = f(4x+1) = 2(4x+1)^2 + 5(4x+1)$

(B) $h(x) = f \circ g(x) = \sqrt{1-x^2}$: Here we are taking the square root of a quadratic

$\Rightarrow g(x) = (1-x^2)$; $f(x) = \sqrt{x}$

$\Rightarrow h(x) = f \circ g(x) = f(1-x^2) = \sqrt{1-x^2}$

The reason for the previous exercise is to identify a function that is expressible as a composition of others, so that we may take the derivative in terms of the composition

The Chain Rule

If $h(x) = f \circ g(x) = f(g(x))$, and both $f'(x), g'(x)$ exist, then we may find the derivative $h'(x)$ as

$$h'(x) = g'(x) \cdot f'(g(x))$$

Chain Rule (Alternate Form)

If y is a function of u, $y = f(u)$ and if u is a function of x, say $u = g(x)$ then $y = f(u) = f(g(x))$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- Here we can remember the chain rule by pretending that $\frac{dy}{du}$ and $\frac{du}{dx}$ are fractions with du "cancelling out"

e.g. Find the derivative using the chain rule where needed

(A) $D_x\left[8x^4 - 5x^2 + 1\right]^4$ If we think of $f(x)$ as $f(x) = x^4$, we can use the chain rule instead of foiling 4 times!

$$\Rightarrow D_x\left[8x^4-5x^2+1\right]^4 = 4\left[8x^4-5x^2+1\right]^{4-1} \cdot \frac{d}{dx}\left[8x^4-5x^2+1\right]$$

$$= 4\left[8x^4-5x^2+1\right]^3\left[32x^3-10x\right]$$

(B) $\Psi(t) = -6t\,(5t^4-1)^4$ Now we need the product rule and chain rule!

$$\Psi'(t) = -6t\left[4(5t^4-1)^3(20t^3)\right] + (-6)(5t^4-1)^4 = -6\left[(5t^4-1)^3(80t^4)\right] + (-6)(5t^4-1)$$

$$= -6(5t^4-1)^3\left[80t^4 + (5t^4-1)\right] = -6(5t^4-1)^3\left[85t^4-1\right]$$

(c) $\xi(t) = \dfrac{(5t-6)^4}{3t^2+4}$ Here, we need the quotient rule and chain rule

$$\xi'(t) = \frac{(3t^2+4)\left[4(5t-6)^3(5)\right] - (5t-6)^4\left[6t\right]}{\left[3t^2+4\right]^2}$$ Now factor the numerator

$$\Rightarrow \xi'(t) = \frac{2(5t-6)^3\left[10(3t^2+4) - (5t-6)(3t)\right]}{\left[3t^2+4\right]^2} = \frac{2(5t-6)^3\left[30t^2+40-15t^2+18t\right]}{\left[3t^2+4\right]^2}$$

$$= \frac{2(5t-6)^3\left[15t^2+18t+40\right]}{\left[3t^2+4\right]^2}$$

Using the chain rule, we can easily prove the quotient rule

Pf of Quotient Rule: Suppose $f(x) = \dfrac{u(x)}{v(x)} = u(x)\left[v(x)\right]^{-1}$. Then, by the product and chain rule:

$$f'(x) = u'(x)\left[v(x)\right]^{-1} + u(x)\left[-1(v(x))^{-2}v'(x)\right] = \frac{u'(x)}{v(x)} - \frac{u(x)v'(x)}{\left[v(x)\right]^2}$$

common denominator $[v(x)]^2$ $\Rightarrow$ $\boxed{f'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{\left[v(x)\right]^2}}$

e.g. Find the derivative using the product and chain rule (Not quotient rule!)

$$f(x) = \frac{2x-1}{4x+3} = (2x-1)(4x+3)^{-1}, \text{ then}$$

$$f'(x) = (2x-1)\left[-1(4x+3)^{-2}(4)\right] + 2(4x+3)^{-1} = \frac{-4(2x-1)}{(4x+3)^2} + \frac{2}{(4x+3)}$$

$$= \frac{(-8x+4) + 2(4x+3)}{(4x+3)^2} = \frac{\cancel{-8x} + \cancel{8x} + 4 + 6}{(4x+3)^2} = \boxed{\frac{10}{(4x+3)^2}}$$

§4.4: Derivatives of Exponential Functions

What is the derivative of the exponential function: $f(x) = e^x$?

<u>Derivative of e^x:</u>

$$\boxed{\frac{d}{dx} e^x = e^x}$$

So what is the derivative of a^x, for any $a \in \mathbb{R}$ with $a > 0$, $a \neq 1$

e.g.) Observe

$$a^x = e^{\ln a^x} = e^{x \ln a}, \text{ then by the chain rule:}$$

$$\frac{d}{dx}[a^x] = \frac{d}{dx}\left[e^{x \ln a}\right] = (\ln a) e^{x \ln a} = (\ln a) e^{\ln a^x} = (\ln a) a^x$$

<u>Derivative of a^x; $a > 0$, $a \neq 1$:</u>

$$\boxed{\frac{d}{dx}[a^x] = (\ln a) a^x}$$

<u>Derivative of $a^{g(x)}$ and $e^{g(x)}$:</u> (By the chain rule)

$$\frac{d}{dx}\left[a^{g(x)}\right] = (\ln a)\left(a^{g(x)}\right) \cdot g'(x)$$

$$\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)} \cdot g'(x)$$

e.g. Find the derivative:

(A) $y = -8e^{3x} \Rightarrow \frac{dy}{dx} = (-8)(e^{3x}) \cdot \frac{d}{dx}(3x) = (-8)e^{3x} \cdot 3 = -24e^{3x}$

(B) $s = 4^{-5t+2} \Rightarrow \frac{ds}{dt} = \left[(\ln 4)(4^{-5t+2})\right] \cdot -5 = (-5\ln 4)\, 4^{-5t+2}$

(C) $\psi = -3e^{3x^2+5} \Rightarrow \frac{d\psi}{dx} = \left(-3e^{3x^2+5}\right) \cdot \frac{d}{dx}(3x^2+5) = (-18x)e^{3x^2+5}$

(D) $y = \underbrace{(3x^3-4x)}_{(u)}\underbrace{e^{-5x}}_{(v)} \xRightarrow[\text{rule}]{\text{product}} y' = uv' + u'v = (3x^3-4x)(-5e^{-5x}) + (9x^2-4)e^{-5x}$

$= e^{-5x}\left[-15x^3 + 20x + 9x^2 - 4\right]$

(E) $f(t) = (e^{t^2} + 5t)^3 \Rightarrow f'(t) = 3(e^{t^2}+5t)^2 \cdot \frac{d}{dt}\left[e^{t^2}+5t\right]$

$= \boxed{3(e^{t^2}+5t)^2 \cdot (2te^{t^2}+5)}$

(F) $y = \frac{x^2}{e^x} = \underbrace{x^2}_{u}\underbrace{e^{-x}}_{v} \Rightarrow \frac{dy}{dx} = uv' + u'v = -x^2e^{-x} + 2xe^{-x}$

$= e^{-x}\left[-x^2 + 2x\right]$

$= xe^{-x}\left[2 - x\right]$

(G) $y = \frac{e^x + e^{-x}}{x} = \frac{u(x)}{v(x)}$

$\xRightarrow[\text{rule}]{\text{quotient}} \frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} = \frac{x\left[e^x - e^{-x}\right] - \left[e^x + e^{-x}\right](1)}{x^2}$

§4.5: Derivatives of Logarithmic Functions

What is the derivative of a logarithmic function? $f(x) = \log_a(x)$

i.e. $a^{f(x)} = x$

Since both sides of the equation are equal, lets take the derivative

$$\Rightarrow \frac{d}{dx}\left[a^{f(x)}\right] = \frac{d}{dx}[x] \Rightarrow (\ln a) f'(x)\, a^{f(x)} = 1$$

$(\because a^{f(x)} = x)$

$$\Rightarrow (\ln a) f'(x)\, x = 1 \Rightarrow f'(x) = \frac{1}{x \ln a}$$

Derivative of $\log_a x$

$$\frac{d}{dx}\left[\log_a x\right] = \frac{1}{\ln a \cdot x} \qquad (a>0,\ a\neq 1)$$

Now when $a = e \Rightarrow \ln a = \ln e = 1$ so...

Derivative of $\ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

e.g.) Find the derivative:

(A) $y = \ln 8x \Rightarrow \frac{dy}{dx} = \frac{1}{8x} \cdot \frac{d}{dx}[8x] = \frac{1}{8x} \cdot 8 = \boxed{\frac{1}{x}}$

(B) $\psi(x) = \ln(1+x^3) \Rightarrow \psi'(x) = \frac{1}{(1+x^3)} \cdot \frac{d}{dx}[1+x^3] = \boxed{\left(\frac{1}{1+x^3}\right) 3x^2}$

(C) $\xi(t) = \ln\sqrt{t+5}$

$$\Rightarrow \xi'(t) = \frac{1}{\sqrt{t^2+5}} \cdot \frac{d}{dt}\left(\sqrt{t^2+5}\right) = \frac{1}{\sqrt{t^2+5}} \cdot \frac{1}{2}(t^2+5)^{-1/2} \cdot 2t = \boxed{\frac{t}{t^2+5}}$$

Here's a formula if the chain rule still freaks you out...

Logarithm Derivative Rules

$$\boxed{\begin{array}{ll} D_x\left[\log_a|x|\right] = \dfrac{1}{\ln a \cdot x} \; ; & D_x\left[\log_a|g(x)|\right] = \dfrac{g'(x)}{(\ln a)\, g(x)} \\ D_x\left[\ln x\right] = \dfrac{1}{x} \; ; & D_x\left[\ln|g(x)|\right] = \dfrac{g'(x)}{g(x)} \end{array}}$$

e.g.) Find the derivative:

(A) $y = \ln(5x^3 - 2x)^{3/2}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(5x^3-2x)^{3/2}} \cdot \frac{3}{2}(5x^3-2x)^{1/2}(15x^2-2)$$

$$= \left[\frac{3}{2}(15x^2-2)\right]\frac{(5x^3-2x)^{1/2}}{(5x^3-2x)^{3/2}} = \frac{3}{2}\left[\frac{(15x^2-2)}{(5x^3-2x)^{1}}\right]$$

(B) $y = \underset{u}{e^{x^2}}\,\underset{v}{\ln x}$

$$\xRightarrow{\text{product rule}} y'(x) = uv' + u'v = \left(e^{x^2} \cdot \frac{1}{x}\right) + \left(2xe^{x^2}\right)\ln x$$

(C) $y = \dfrac{e^x}{\ln x} = \dfrac{u}{v} \xRightarrow{\text{quotient rule}} y' = \dfrac{vu' - uv'}{v^2} = \dfrac{(\ln x)e^x - e^x\left(\frac{1}{x}\right)}{(\ln x)^2} \cdot \left(\dfrac{x}{x}\right)$ simplify

$$= \frac{xe^x \ln x - e^x}{x(\ln x)^2}$$

(D) $\alpha(t) = \left(e^{2t} + \ln t\right)^3$

$$\Rightarrow \alpha'(t) = 3\left(e^{2t} + \ln t\right)^2 \left(2e^{2t} + \frac{1}{t}\right)$$

(E) $\zeta(x) = \sqrt{e^{-x} + \ln 2x} = \left(e^{-x} + \ln 2x\right)^{1/2}$

$2 \cdot \frac{1}{2x} = \frac{1}{x}$

$$\Rightarrow \zeta'(x) = \frac{1}{2}\left(e^{-x} + \ln 2x\right)^{-1/2} \left[-e^{-x} + \frac{1}{\cancel{2}x}\right]$$