§1.1: Slopes and Equations of Lines

When describing 2 measureable quantities we may be interested in how they are mathematically related. Often we write their relationship in an equation that shows how each variable changes with respect to one-another.

For example, we may be interested in how to compute the interest a bank account yields in a year.

e.g.) Suppose a bank account pays 6% simple interest per year. Then we can calculate the interest, I, that is accumulated depending on the principal in the account, P. We can write this relationship with an equation to find the interest given a certain principal deposit as:

$$I = (.06)P$$

since we accumulate $6\% = .06$ of the principal as interest.

This relationship can help us calculate the interest depending on our deposit or the amount of the deposit given interest collected

$$I = (.06)P \iff P = I/.06$$

For instance if we deposit \$100 then $P = (\$100)$ we find

$$I = (.06)100 = \$6$$

We can describe this data point as an ordered pair

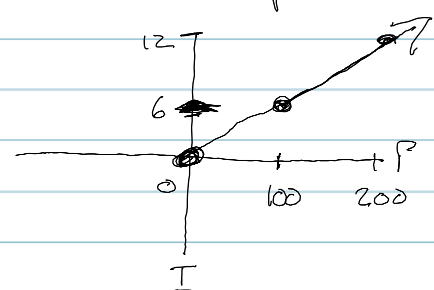
$(100, 6)$ where the first value represents the principal, or independent variable and the second term represents the interest obtained

$$(100, 6) = (P, I)$$

We can plot each of these ordered pairs on a Cartesian coordinate system. Plotting the full set of points is called the graph of the equation.

Now from $I = (.06)P$ we can write down a few ordered pairs to help us visualize the relationship.

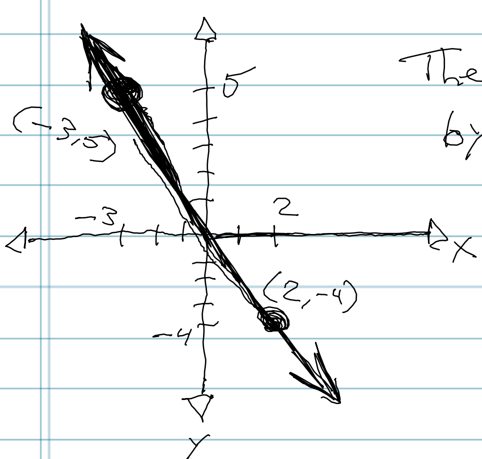
(P, I)
(0, 0)
(100, 6)
(200, 12)



Here all the data points lie in the 1^{st} quadrant since negative values for principal and interest do not make sense

An important characteristic of any line is where it touches the X-axis and y-axis (Above the x-axis is the Principal (P) and y-axis is interest I) These points are called the X-intercept and y-intercept (resp.) In the above example the x and y intercepts occur at the same point (0,0).

Now consider another example of a line plotted on the x-y axis.



The slope or steepness of a line can be found by measuring the change in both the X and Y directions. Any 2 pts on a line will be enough to determine slope.

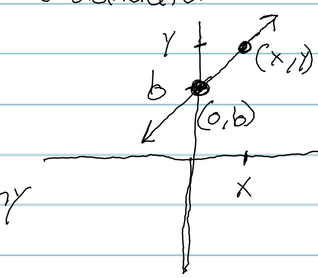
$$M = \frac{\text{change in } Y}{\text{change in } X} = \frac{\Delta Y}{\Delta X}$$

For the line depicted above we can measure the slope using the 2 points given: $\underset{(x_1, y_1)}{(-3,5)}$, $\underset{(x_2, y_2)}{(2,-4)}$

$$\Delta y = y_2 - y_1 = -4 - 5 = -9$$
$$\Delta x = x_2 - x_1 = 2 - (-3) = 5$$
$$\Rightarrow \quad M = \frac{\Delta y}{\Delta x} = -\frac{9}{5}$$

Notice the slope is negative so the line moves "downward" from left to right

Using our equation for the slope of a line we can generalize a standard equation for a line.

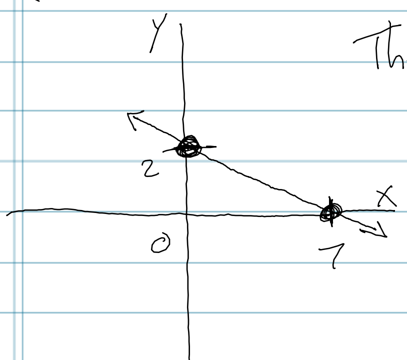


Suppose a line has slope m and y intercept b, i.e. the line passes through $(0,b)$. Suppose (x,y) is any other point on the line. Then, from our slope equation:

$$m = \frac{y-b}{x-0} \iff mx = y - b \iff \boxed{y = mx + b}$$ "Slope Intercept Form"

Now we can find the equation of any line (with a defined slope) as long as we know the slope and y-intercept

e.g.) Find equation of the line given:



The 2 points displayed are $(0,2)$, $(7,0)$
we see immediately that the y-intercept is $(0,2)$

$$\Rightarrow b = 2$$

Now we find the slope

$$m = \frac{0-2}{7-0} = -\frac{2}{7} \Rightarrow \boxed{y = -\frac{2}{7}x + 2}$$

Now, suppose we don't know the y-intercept. Suppose a line passes through the points (x_1, y_1) and (x,y) and has slope m. Then:

$$m = \frac{y-y_1}{x-x_1} \iff \boxed{(y-y_1) = m(x-x_1)}$$ "Point Slope Form"

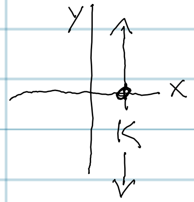
e.g.) Find the equation of a line with a slope of $5/4$ that passes through the point $(3,-7)$.

Pt Slope Form $\Rightarrow$ $y - (-7) = \frac{5}{4}(x-3)$

$$\iff y = \frac{5}{4}x - \frac{15}{4} - 7 = \frac{5}{4}x - \frac{43}{4}$$

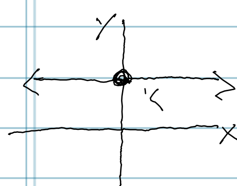
Special Cases:

Vertic Line: $\boxed{x=k}$ for some $k \in \mathbb{R}$. This means that the slope is undefined, since in our slope equation



$m = \frac{\Delta y}{\Delta x}$; Δx (i.e. the change in x) is 0. $\Rightarrow m = \varnothing$

Horizontal Line: $\boxed{y=k}$ for some $k \in \mathbb{R}$. This type of line has 0 slope, $m=0$, since there is no change in y for any points on the line, $\Delta y = 0$



$\Rightarrow m = \frac{\Delta y}{\Delta x} = \frac{0}{\Delta x} = 0$

Properties: Given two lines, $y_1 = m_1 x + b_1$; $y_2 = m_2 x + b_2$

$y_1 \parallel y_2$ 1. Two lines are parallel $\Leftrightarrow$ they have the same slope ($m_1 = m_2$) or they are both vertical (undefined slope)

$y_1 \perp y_2$ 2. Two lines are perpendicular $\Leftrightarrow$ the product of their slopes is -1 ($m_1 \cdot m_2 = -1$) or if one line is vertical and the other is horizontal

(Note $m_1 m_2 = -1 \Rightarrow m_1 = \frac{-1}{m_2}$)

e.g. Find the equation of the line passing through $(3,5)$ and parallel to $2x+5y=4$. Then find a line passing through $(3,5)$ that is perpendicular

First rewrite the eqn in slope intercept form: $2x+5y=4 \Leftrightarrow y = \frac{-2}{5}x + \frac{4}{5}$

$\Rightarrow m_0 = \frac{-2}{5}$

Parallel ($m = m_0$)

Point Slope Form $\Rightarrow y - 5 = \frac{-2}{5}(x-3)$

$\Leftrightarrow y = \frac{-2}{5}x + \frac{6}{5} + 5$

$\Rightarrow \boxed{y = \frac{-2}{5}x + \frac{31}{5}}$

Perpendicular ($m \cdot m_0 = -1 \Rightarrow m = \frac{-1}{m_0} = \frac{5}{2}$)

$\Rightarrow y - 5 = \frac{5}{2}(x-3)$

$\Rightarrow y = \frac{5}{2}x - \frac{15}{2} + 5$

$\Rightarrow \boxed{y = \frac{5}{2}x - \frac{5}{2}}$

§1.2 : Linear Functions / Applications

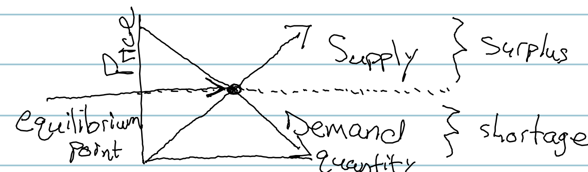
When describing a relationship between two variables, its common to use <u>functional notation</u>. Usually use letters as function names.
We've already described linear functions:

$$y = f(x) = mx + b \; ; \quad m, b \in \mathbb{R} \qquad \text{"Linear Function"}$$

e.g. Given $g(x) = -4x + 5$ find $g(0)$, $g(b)$
Plug and Chug:

$g(0) = -4(0) + 5 = 5$; $g(b) = -4(b) + 5$

<u>Supply and Demand</u>:



In economics, price trends of an item can be estimated based on both the supply and demand for the item. The supply and demand functions can be approximated linearly.
- Generally, as the price of an item increases, consumers are less likely to buy, so demand derseses ($P\uparrow \Rightarrow D\downarrow$)
- However while price increases, producers tend to create more product, increasing supply
- When there is too little demand to meet the supply, we have a <u>surplus</u>. Prices will then decrease to adjust.
- Over time, prices tend toward an equilibrium point, where there is an exact supply to meet the demand ($S = D$)

e.g. Suppose the demand ~~for~~ and the price of strawberries are related as:

$P = D(q) = 5 - .25q$ where P is price in \$ and q is quantity ~~supplied~~ demanded (in hundreds of quarts)

(A) Find the price per level of demand: 0 quarts, 400 quarts?
Plug into our function w/ $q = 0, 400$

q measured in hundreds of quarts!

0 quarts: $P = D(0) = 5 - 0 = \$5$; 400 quarts: $P = D(400) = 5 - \frac{4}{4} = 5 - 1 = \$4$

(B) Find the quantity demanded at each price: \$4.50, \$3.25

Now we have our price, so we are looking for q

$\$4.50 = 5 - \frac{1}{4}q$

$\Leftrightarrow q = (5 - 4.50)4 = \frac{1}{2}(4) = 2$

$\Rightarrow \boxed{200 \text{ quarts}}$

$\$3.25 = 5 - \frac{1}{4}q$

$\Leftrightarrow q = 4(5 - 3.25) = 4(1\frac{3}{4})$

$= 4(\frac{7}{4}) = 7 \Rightarrow \boxed{700 \text{ quarts}}$

Suppose the price and supply of strawberries are related by

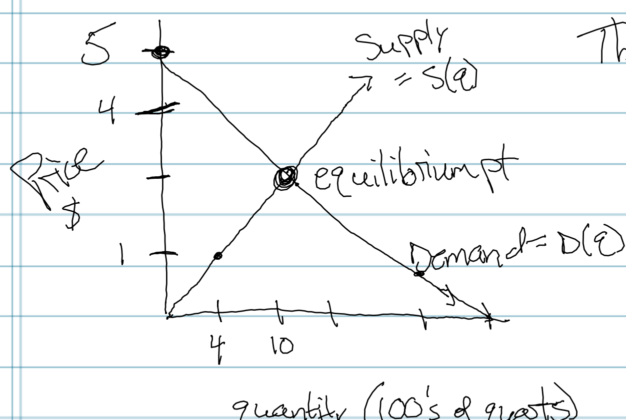
$p = S(q) = .25q = q/4$ where p is the price in \$\$ and q the quantitized supplied in 100's of quarts

(C) Find the quantity given each price: \$0, \$2

$p = \$0 = .25q \Rightarrow q = 0$ quarts ;

$p = \$2 = .25q \Rightarrow q = 8 \Rightarrow \boxed{800 \text{ quarts}}$

(D) Plot the supply and demand functions and find the equilibrium point.



The equilibrium point occurs when the supply and demand functions intersect:

$$D(q) = S(q)$$

$\Rightarrow 5 - \frac{1}{4}q = \frac{1}{4}q$

$\Rightarrow \frac{1}{2}q = 5 \Rightarrow q = 10 \Rightarrow \boxed{1000 \text{ quarts}}$

$\Rightarrow p = D(10) = S(10) = \$\frac{10}{4} = \boxed{\$2.50}$

In practice, may not have perfectly linear supply and demand functions. Usually data is taken and if the relationship appears approximately linear, a technique called <u>linear regression</u> may determine the best fit line.

Cost Analysis

- The cost of manufacturing an item usually consists of (design, worke
  that generally doesn't change as more items are produced, and
  a cost per item (labor, materials, shipping)
- when cost is of this form it can be modeled with a linear equation:
  $C(x) = mx + b$; where m is the marginal cost and b the fixed cost.

- The marginal cost is approximately the cost of producing an additional item. This is the rate of change of the cost function. When the cost function is linear, this is simply the slope, m, of $C(x) = mx + b$. (since a line has a constant rate of change)

- The revenue is the payoff after selling x units:
  $R(x) = px$ where p is the price of 1 unit

- The profit is simply the difference between the cost and revenue
  $P(x) = R(x) - C(x)$

- Profit only occurs when revenue exceeds cost. The break even quantity occurs when cost equals revenue
  $$R(x) = C(x)$$

e.g. The cost of producing a flu shot is approximately \$10. Suppose it costs \$1500 to create 100 batches and that the cost function is linear: $C(x) = mx + b$.

(A) Find the cost function $C(x)$:

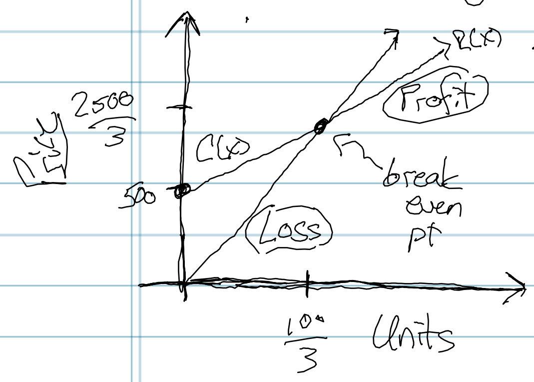
The marginal cost (cost of producing each shot) is given as $\$10 = m$
Since we also know how ~~to creat~~ much 100 batches cost, we have a data point $(100, 1500)$. Use point slope form:

$$C(x) - 1500 = 10(x - 100) \Rightarrow \boxed{C(x) = 10x + 500}$$

$\Rightarrow$ fixed cost is 500

(8) Suppose flu shots are being sold for \$25. How many shots must be sold for the pharmacy to break even.

The revenue is given by $R(x) = px = 25x$ (since sold at \$25 per shot)



The break even point occurs when $C(x) = R(x)$

$$C(x) = R(x) \Rightarrow 10x + 500 = 25x$$

$$\Rightarrow 15x = 500 \Rightarrow \boxed{x = \frac{500}{15} = \frac{100}{3}}$$

$$\Rightarrow C\left(\tfrac{100}{3}\right) = R\left(\tfrac{100}{3}\right) = 25\left(\tfrac{100}{3}\right) = \boxed{\frac{2500}{3}}$$

e.g.) Suppose we wanted to find the relationship between the Celcius and Fahrenheit Temperature Scale. We know the relationship is linear. We also know where water boils and freezes on each scale:

~~Freezing~~ Boiling Pt $(100°C, 212°F)$; Freezing Pt $(0°C, 32°F)$

Since the equation is linear we need the slope:

$$m = \frac{\Delta y}{\Delta x} = \frac{212 - 32}{100 - 0} = \frac{9}{5}$$

Now, we know the y-intercept from the Freezing Point:

Slope-intercept $\Longrightarrow$ $\boxed{F = \frac{9}{5}C + 32}$

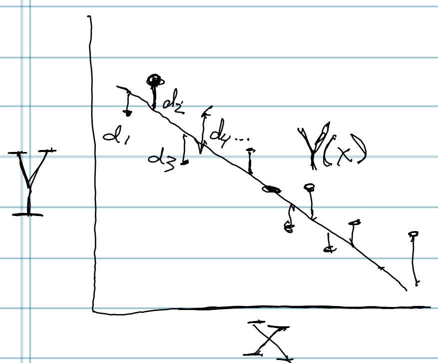
$\Longleftrightarrow$ $\boxed{C = \frac{5}{9}(F - 32)}$

§1.3: The Least Squares Line

Often when we are comparing 2 different quantitative variables, we try to form an equation that can approximate the data set. This model (in most cases) may not be perfect, but can describe the trend we see in the data.

When data appears linear (i.e. we can almost draw a line through the data), we use <u>linear regression</u> to fit a "least squares line" to the dataset. The least squares line is created by minimizing the sum of squares of vertical distances from pts on the line (depicted as d_i)

<u>Least Squares Line</u>: The least squares line $y = mx + b$ that gives the best fit to the data points $(x_1, y_1) \ldots (x_n, y_n)$ has slope m, y intercept, b given by:



$$m = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{n\left(\sum x^2\right) - \left(\sum x\right)^2}$$

$$b = \frac{\sum y - m\left(\sum x\right)}{n}$$

This yields a compact equation, that can approximate our data set.

<u>Warning!</u>: Don't assume all data can be approximated by linear regression. If the data trends are not approximately linear, then this equation tells you NOTHING!

We can use a measure to determine how accurate the least squares line approximates the data, called the <u>correlation coefficient, r</u>:

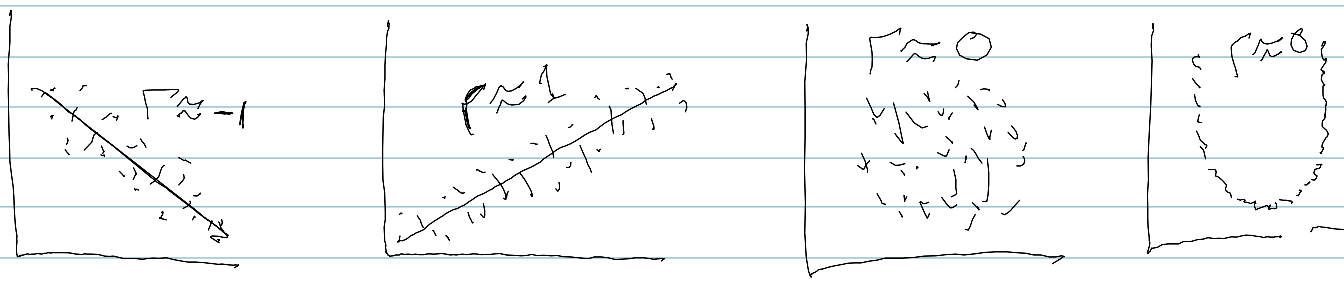
$$r = \frac{n\left(\sum xy\right) - \left(\sum x\right)\left(\sum y\right)}{\sqrt{n\left(\sum x^2\right) - \left(\sum x\right)^2} \cdot \sqrt{n\left(\sum y^2\right) - \left(\sum y\right)^2}}$$

These equations look pretty hefty, but don't worry computers have plenty of software that can calculate the least squares line and correlation coefficient with ease. Just know WHY you would use this method and WHAT it tells you.

Now correlation will give you a # between $[-1,1]$

$r \in [-1,1]$ with $r=-1$ or 1 means perfect correlation (i.e. all data pts fall on the line)

When $r \approx 0$ we have No correlation, so a linear model is NOT appropriate to fit the data.



The square of the correlation coefficient tells us the fraction of variation in y that is explained by the linear relationship between x and y.

i.e.) if $r = -0.963 \Rightarrow r^2 = 0.927$ then we say 92.7% of the variation in y is explained by the least squares line and the other 7.3% is due to error

We define the error to be the distance between a data point and its corresponding value on the least squares line. (depicted as $d_1, d_2, \ldots$ on the picture of the least squares line)