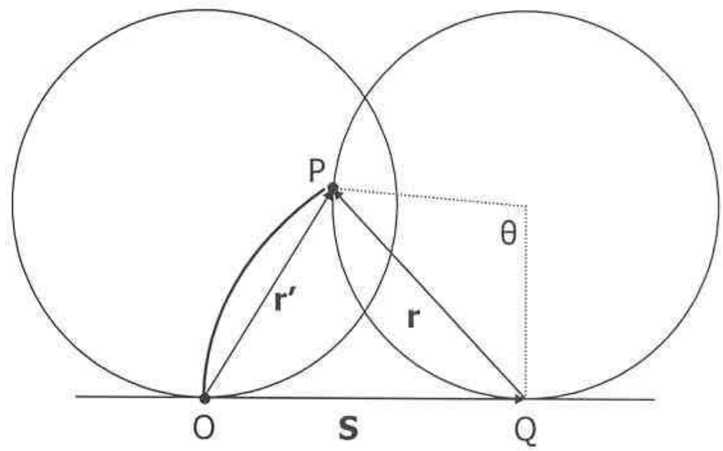
The Tautochronous Pendulum.

Let the cylinder roll without slipping on the horizontal surface from initial point of contact O to new point of contact Q. Let O be the fixed origin of the coordinate system. The initial point of contact at O has moved to a new position P. The coordinates of that point are the cycloid coordinates:

$x = a\theta - a\sin\theta$ (1)

$y = a - a\cos\theta$ (2)

where a is the radius of the circle.



Proposition I: Segment QP is perpendicular to the cycloid line at P.

Proof

We need show that the line element $d\mathbf{s}$ that is tangent to the cycloid at P is perpendicular to QP. Consider vectors **S**, **r** and **r'** as shown in the drawing. Clearly, $\mathbf{r} = \mathbf{r'} - \mathbf{S}$. Then,

$\mathbf{r'} = (a\theta - a\sin\theta)\,\mathbf{e}_x + a\,(1 - \cos\theta)\,\mathbf{e}_y$

$\mathbf{S} = a\theta\,\mathbf{e}_x$

$\mathbf{r} = \mathbf{r'} - \mathbf{S} = -a\sin\theta\,\mathbf{e}_x + a\,(1 - \cos\theta)\,\mathbf{e}_y$

Now

$d\mathbf{s} = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y = a\,(1 - \cos\theta)\,d\theta\,\mathbf{e}_x + a\sin\theta\,d\theta\,\mathbf{e}_y$

and,

$d\mathbf{s}\cdot\mathbf{r} = -a^2\sin\theta\,(1 - \cos\theta)\,d\theta + a^2\sin\theta\,(1 - \cos\theta)\,d\theta = 0.$

Therefore, $d\mathbf{s}$ is perpendicular to QP at P, QED.

Proposition II: The length of arc segment OP along the cycloid is $s = 4a(1 - \cos\frac{\theta}{2})$

Proof

$ds = \sqrt{(dx)^2 + (dy)^2}$

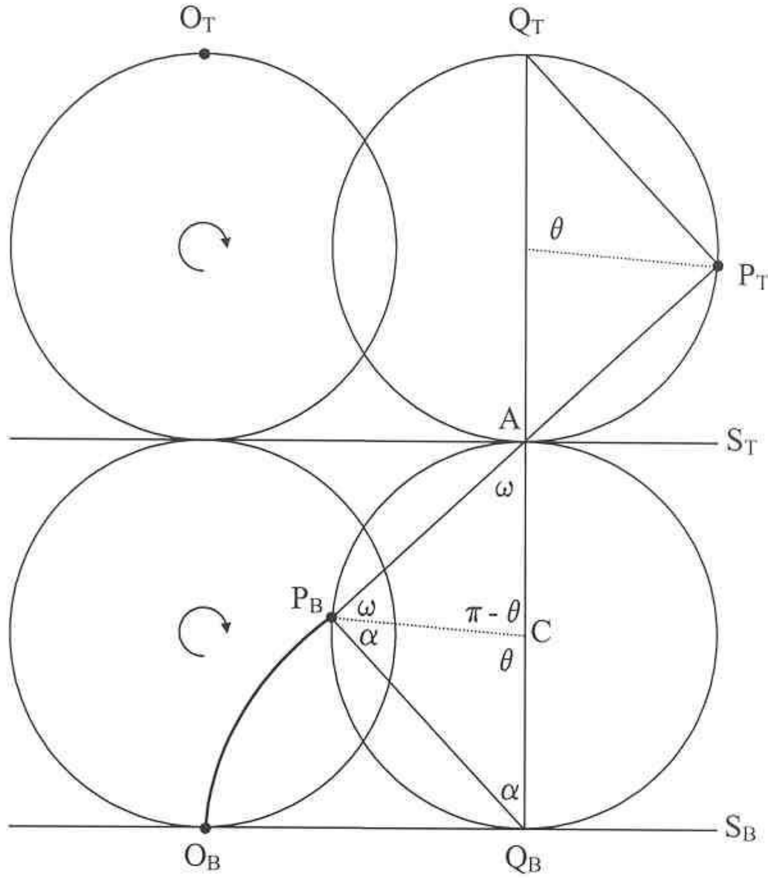
$dx = (1 - \cos\theta)a\,d\theta; \quad dy = a\sin\theta\,d\theta$

$ds = a\,d\theta\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2} = a\,d\theta\sqrt{2(1-\cos\theta)} = 2a\sin\frac{\theta}{2}\,d\theta$

$s = 2a\int_0^\theta \sin\frac{\theta}{2}\,d\theta = 4a\left[-\cos\frac{\theta}{2}\right]_0^\theta = 4a\left(1 - \cos\frac{\theta}{2}\right)$

QED.

Proposition III: Consider two identical cylinders, bottom and top. They roll without slipping on bottom and top surfaces S_B and S_T with identical velocities to the right. Let O_B and O_T be cycloid generator points for each cylinder. **When the cylinders have rotated by angle θ, line segment P_TP_B is tangent to the cycloid at point P_B.**



Proof:

We need to show that segment P_TP_B is perpendicular to segment Q_BP_B at point P_B. Then, by Proposition I, segment P_TP_B is tangent to the cycloid at point P_B.

First we note that, by symmetry, triangles AQ_BP_B and AQ_TP_T are congruent. This means that line P_TP_B has to pass through point A, the point at which the cylinders are tangent to one another.

From isosceles triangle $\triangle C P_TP_B$ we have

$$2\alpha + \theta = \pi \quad \rightarrow \quad \alpha = \frac{\pi}{2} - \frac{\theta}{2}$$

and from isosceles $\triangle CAP_B$

$$2\omega + (\pi - \theta) = \pi \quad \rightarrow \quad \omega = \frac{\theta}{2}$$

Adding the two equations gives $\alpha + \omega = \frac{\pi}{2}$ which proves the proposition.

It follows from the above that segment P_TP_B is perpendicular to the tangent of the top cycloid at point P_T.