Math 255 - Spring 2022
Solving multiple linear congruences
15 points

This homework invites you to solve several systems of linear congruences. Students taking this class for graduate credit are only required to answer problems 4 and 5 (though they are welcome to answer all problems of course).

1. (6 points) Solve each of the following systems of congruences. For each system of equations, be sure to list **all** distinct solutions, and the ring they belong to.
   (a) $x \equiv 5 \pmod{6}, \quad x \equiv 4 \pmod{11}, \quad x \equiv 3 \pmod{17}$
   (b) $x \equiv 2 \pmod{5}, \quad 2x \equiv 3 \pmod{7}, \quad 3x \equiv 4 \pmod{11}$
   (c) $4x \equiv 4 \pmod{8}, \quad 5x \equiv 6 \pmod{25}, \quad 3x \equiv 6 \pmod{27}$
2. (2 points) Find a multiple of 7 that leaves a remainder of 1 when divided by $2, 3, 4, 5$ and 6.
3. (2 points) Solve the linear congruence $17x \equiv 3 \pmod{210}$ by noticing that $210 = 2 \cdot 3 \cdot 5 \cdot 7$ and solving the system
$$17x \equiv 3 \pmod{2}, \quad 17x \equiv 3 \pmod{3}$$
$$17x \equiv 3 \pmod{5}, \quad 17x \equiv 3 \pmod{7}.$$
4. (2 points) Consider the system of congruences
$$6x \equiv 3 \pmod{9}$$
$$10x \equiv 8 \pmod{16}.$$
   (a) Solve each of the following congruences:
      i. $6x \equiv 3 \pmod{9}$
      ii. $10x \equiv 8 \pmod{16}$
   (b) The system of congruences above is equivalent to 6 distinct systems of congruences of the form $x \equiv a_1 \pmod{9}, \quad x \equiv a_2 \pmod{16}$. Write down these 6 systems and solve each of them using the Chinese Remainder Theorem.
   (c) The system of congruences above is equivalent to a single system of congruences of the form $a_1 x \equiv b_1 \pmod{3}, \quad a_2 x \equiv b_2 \pmod{8}$. Write down this system, solve it using the Chinese Remainder Theorem, and lift your solutions to $\mathbb{Z}/144\mathbb{Z}$.
5. (3 points) Let x, r, s and m be integers, with $m > 1$. Show that if $x \equiv r \pmod{m}$ and $x \equiv s \pmod{m+1}$, then
$$x \equiv r(m+1) - sm \pmod{m(m+1)}.$$