Name:

Problem 1: *Let K be the field with $81 = 3^4$ elements.*

a) How many nonzero elements of K generate $K^\times$ multiplicatively?

b) How many elements of K are primitive elements for the extension $K/\mathbb{F}_3$?

Solution:

a) $K^\times$ is a cyclic group with $81-1 = 80$ elements. It therefore has $\varphi(80)$ generators, and

$$\varphi(80) = \varphi(16)\varphi(5) = (16-8)\cdot 4 = 8\cdot 4 = 32.$$

b) $\alpha \in K$ is a primitive element of $K/\mathbb{F}_3$ if and only if it is a root of an irreducible polynomial of degree 4 over $\mathbb{F}_3$. We know that $x^{81} - x$ is the product of all irreducible polynomials over $\mathbb{F}_3$ of degrees 1, 2 and 4 (this is because 1, 2 and 4 are all of the divisors of 4). Similarly, $x^9 - x$ is the product of all irreducible polynomials over $\mathbb{F}_3$ of degrees 1 and 2. Therefore

$$\frac{x^{81}-x}{x^9-x}$$

is the product of all irreducible polynomials of degree 4. The roots of this polynomial are exactly all of the primitive elements for this extension, and since this polynomial is separable, it has $81 - 9 = 72$ roots.