Math 295 - Spring 2020
Homework 8

This homework is due on Wednesday, March 4. All problems are adapted from Munkres's *Topology*.

For the first two problems you will need the following definition: Let $X$ be a topological space and $A \subset X$ be a subspace. We define the *boundary* of $A$ by the equation

$$\operatorname{Bd} A = \overline{A} \cap \overline{(X - A)}.$$

1. (a) Show that
$$\operatorname{Int} A \cap \operatorname{Bd} A = \varnothing,$$
where $\operatorname{Int} A$ is the interior of $A$.
   (b) Show that
$$\overline{A} = \operatorname{Int} A \cup \operatorname{Bd} A.$$
   (c) Show that $\operatorname{Bd} A = \varnothing$ if and only if $A$ is both open and closed.
   (d) Show that $U$ is open if and only if $\operatorname{Bd} U = \overline{U} - U$.
   (e) Show that if $U$ is open, then $U \subset \operatorname{Int}(\overline{U})$. Prove that the containment is not an equality in general by showing that if $U = (0, 1) \cup (1, 2) \subset \mathbb{R}$ (where $\mathbb{R}$ has the standard topology), then $U \neq \operatorname{Int}(\overline{U})$.

2. Find the boundary and interior of the following subsets of $\mathbb{R}^2$, where $\mathbb{R}$ has the standard topology, and $\mathbb{R}^2$ has the product topology:
   (a) $A = \{x \times y \mid y = 0\}$
   (b) $B = \{x \times y \mid x > 0 \text{ and } y \neq 0\}$

3. Suppose that $f \colon X \to Y$ is continuous. If $x$ is a limit point of the subset $A$ of $X$, is it necessarily true that $f(x)$ is a limit point of $f(A)$? To support your answer, either prove that this statement is true, or give a counterexample.

For the last two problems you will need the following definition: Let $F \colon X \times Y \to Z$, where $X, Y$ and $Z$ are topological spaces and $X \times Y$ has the product topology. We say that $F$ is *continuous in each variable separately* if for each $y_0 \in Y$, the function $h \colon X \to Z$ given by $h(x) = F(x \times y_0)$ is continuous, and for each $x_0 \in X$, the function $k \colon Y \to Z$ given by $k(y) = F(x_0 \times y)$ is continuous.

4. Let $F \colon X \times Y \to Z$, with $X, Y, Z$ and $X \times Y$ as above. Show that if $F$ is continuous, then $F$ is continuous in each variable separately.

5. Consider the function $F\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, where $\mathbb{R}$ has the standard topology and $\mathbb{R}\times\mathbb{R}$ has the product topology, given by

$$F(x\times y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{if } x\times y\neq 0\times 0,\\ 0 & \text{if } x\times y=0\times 0.\end{cases}$$

(a) Show that $F$ is continuous is each variable separately.

(b) Compute the function $g\colon \mathbb{R}\to\mathbb{R}$ given by $g(x)=F(x\times x)$.

(c) Show that $F$ is not continuous.

Extra problem for graduate credit:

1. Let $X$ be a topological space and $A\subset X$ be a subspace. Let $f\colon A\to Y$ be continuous, and $Y$ be Hausdorff. Show that if $f$ may be extended to a continuous function $g\colon \overline{A}\to Y$, then $g$ is uniquely determined by $f$.