Math 295 - Spring 2020
Solutions to Homework 10

1. (a) We verify the three axioms:
    1. Nonnegativity: The maximum of a set of nonnegative numbers is nonnegative. Furthermore, the maximum of a set of nonnegative numbers is zero if and only if every number in the set is zero.
    2. Symmetry: This follows from the fact that for each i, $|x_i - y_i| = |y_i - x_i|$.
    3. Triangle inequality: Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n), \mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$, and let k be such that $\rho(\mathbf{x}, \mathbf{z}) = |x_k - z_k|$ (the maximum value is attained since n is finite). Then we have

    $$\rho(\mathbf{x}, \mathbf{z}) = |x_k - z_k| \leq |x_k - y_k| + |y_k + z_k| \leq \rho(\mathbf{x}, \mathbf{y}) + \rho(\mathbf{y}, \mathbf{z}),$$

    where here we have used the triangle inequality for the usual absolute value, and the fact that for each i, $|x_i - y_i| \leq \rho(\mathbf{x}, \mathbf{y})$.

   (b) The topology induced by ρ has as a basis the collection

   $$\mathcal{B} = \{B_\rho(\mathbf{x}, r) \mid \mathbf{x} \in \mathbb{R}^n, r > 0\},$$

   and the usual topology on $\mathbb{R}^n$ has as a basis the collection

   $$\mathcal{B}' = \{(a_1, b_1) \times \cdots \times (a_n, b_n) \mid a_i < b_i \in \mathbb{R}\}.$$

   We use Lemma 13.3 to show that these two bases generate the same topology.
   Let $\mathbf{x} \in \mathbb{R}^n$ and $B = B_\rho(\mathbf{y}, r) \in \mathcal{B}$ be such that $\mathbf{x} \in B$. Then there is $\delta > 0$ such that $B_\rho(\mathbf{x}, \delta) \subset B_\rho(\mathbf{y}, r)$. Let $B' = (x_1 - \delta, x_1 + \delta) \times \cdots \times (x_n - \delta, x_n + \delta)$. Then $\mathbf{x} \in B' \subset B_\rho(\mathbf{x}, \delta) \subset B$.
   Now let $\mathbf{x} \in \mathbb{R}^n$ and $B' = (a_1, b_1) \times \cdots \times (a_n, b_n) \in \mathcal{B}'$ be such that $\mathbf{x} \in B'$. Let $r = \min_{i=1}^n (x_i - a_i, b_i - x_i) > 0$, and let $B = B_\rho(\mathbf{x}, r)$. Then if $\mathbf{y} \in B$, for each i we have $|x_i - y_i| \leq \rho(\mathbf{x}, \mathbf{y}) < r$, so $y_i \in (a_i, b_i)$ and $\mathbf{y} \in B'$. Therefore $\mathbf{x} \in B \subset B'$.

2. Let (X, d) be a metric space, and let $x \neq y \in X$. Then $d(x, y) > 0$ by the properties of a metric. We claim that $B_d(x, \frac{d(x,y)}{2})$ and $B_d(y, \frac{d(x,y)}{2})$ are disjoint, and as they are open sets containing x and y respectively, proving this will complete the proof.
   Let $z \in B_d(x, \frac{d(x,y)}{2})$. By the triangle inequality,

   $$d(z, y) \geq d(x, y) - d(x, z) \geq d(x, y) - \frac{d(x, y)}{2} = \frac{d(x, y)}{2},$$

   therefore, $z \notin B_d(y, \frac{d(x,y)}{2})$, and the two balls are disjoint.

3. (a) All three properties follow from the fact that they hold for all $x_1, x_2, x_3 \in X$, and therefore for all $a_1, a_2, a_3 \in A$.

(b) The subspace topology on A is exactly

$$\mathcal{T} = \{U \cap A \mid U \text{ is open in } X\}.$$

We use Lemma 13.2 to show that the balls $B_{d|_{A\times A}}(a, r)$ for $a \in A$ and $r > 0$ form a basis for this topology.

First, we note that

$$B_{d|_{A\times A}}(a, r) = B_d(a, r) \cap A,$$

and therefore each $B_{d|_{A\times A}}(a, r) \in \mathcal{T}$.

Now it remains to show that for every $a \in A$ and $V \in \mathcal{T}$, there are $b \in A$ and $r > 0$ such that $a \in B_{d|_{A\times A}}(b, r) \subset V$. Since $V \in \mathcal{T}$, there is U open in X such that $V = U \cap A$. Now by the characterization of opens in a metric space, there is $\delta > 0$ such that $B_d(a, \delta) \subset U$. Now we have that $B_{d|_{A\times A}}(a, \delta) = B_d(a, \delta) \cap A$, so $a \in B_{d|_{A\times A}}(a, \delta) \subset V$. This is exactly what we need to apply Lemma 13.2, and so the balls in the restricted metric are indeed a basis for the subspace topology on A.