Spring 2020 - Topology
Exam 2
Solutions

#2 a) X is connected if it does not have a separation.

A separation of X is two sets U, V with $X = U \cup V$ and U, V are open, non empty and disjoint.

b) For $\alpha \in J$ let $B_\alpha = A \cup A_\alpha$

1 B_α is connected $\forall \alpha \in J$ because it is the union of 2 connected sets and $A \cap A_\alpha \neq \phi$

Now we have

2 $A \cup \left(\bigcup_{\alpha \in J} A_\alpha \right) = \bigcup_{\alpha \in J} B_\alpha$

Indeed, if $x \in A \cup (\bigcup_{\alpha \in J} A_\alpha)$ then $x \in A$ OR $x \in A_\alpha$ for some α, so $x \in B_\alpha \subset \bigcup_{\alpha \in J} B_\alpha$

Conversely if $x \in \bigcup_{\alpha \in J} B_\alpha$ then $x \in B_\alpha$ for some α so $x \in A \subset A \cup (\bigcup_{\alpha \in J} A_\alpha)$ OR $x \in A_\alpha \subset A \cup (\bigcup_{\alpha \in J} A_\alpha)$

(3) $\bigcup_{\alpha \in J} B_\alpha$ is connected since it is the union of connected sets and $\bigcap_{\alpha \in J} B_\alpha \supset A$ so it is not empty

#3 a) i. $d(v_1, v_2) = 2$

ii. $d(v_2, v_1) = 2$

iii. $d(v_1, v_5) = 3$

iv. $d(v_2, v_4) = 2$

b) It is a function $d: X \times X \to \mathbb{R}$ that is

① Nonnegative: $d(x,y) \geq 0 \quad \forall x, y \in X$ and

$d(x,y) = 0$ iff $x = y$

② Symmetric: $d(x,y) = d(y,x) \quad \forall x, y \in X$

③ Triangle inequality:

$$d(x,z) \leq d(x,y) + d(y,z) \qquad \forall x, y, z \in X$$

c) ① d is the length of a path, which is nonnegative. A path of length zero goes nowhere, and the shortest path $V_i \to V_i$ has length 0.

② the length of a path is independent of what is the beginning and what is the end of the path

③ If one takes the shortest path from V_i to V_j (which is of length $d(V_i, V_j)$) and then the shortest path from

V_j to V_k (which is of length $d(V_j, V_k)$)
then we get a path from V_i to V_k, which is surely at least as long as the shortest path from V_i directly to V_k

$$d(V_i, V_k) \leq d(V_i, V_j) + d(V_j, V_k)$$

shortest path

path going $V_i \to V_j$ on shortest path, then $V_j \to V_k$ on shortest path.

#4 Suppose for a contradiction that X is not connected, so $X = U \cup V$ is a separation. Then U is open so its complement is finite. At the same time, since V has finite complement, it is infinite, but equal to the complement of U which is infinite, contradiction.